A Variational Approach to Subdivision

Leif Kobbelt

Department of Computer Sciences
University of Wisconsin – Madison
1210 West Dayton Street
Madison, WI 53706-1685

December 11, 1995

Abstract

In this paper a new class of interpolatory refinement schemes is presented which in every refinement step determine the new points by solving an optimization problem. In general, these schemes are global, i.e., every new point depends on all points of the polygon to be refined. By choosing appropriate quadratic functionals to be minimized iteratively during refinement, very efficient schemes producing limiting curves of high smoothness can be defined. The well known class of stationary interpolatory refinement schemes turns out to be a special case of these variational schemes.

1 Introduction

Interpolatory refinement is a very intuitive concept for the construction of interpolating curves or surfaces. Given a set of points $\mathbf{p}_i^0 \in \mathbb{R}^d$ which are to be interpolated by a smooth curve, the first step of a refinement scheme consists in connecting the points by a piecewise linear curve and thus defining a polygon $\mathbf{P}_0 = (\mathbf{p}_0^0, \ldots, \mathbf{p}_{n-1}^0)$.

This initial polygon can be considered as a very coarse approximation to the final interpolating curve. The approximation can be improved by inserting new points between the old ones, i.e., by subdividing the edges of the given polygon. The positions of the new points $\mathbf{p}_{2i+1}^1$ have to be chosen appropriately such that the resulting (refined) polygon $\mathbf{P}_1 = (\mathbf{p}_0^1, \ldots, \mathbf{p}_{2n-1}^1)$ looks *smoother* than the given one in some sense (cf. Fig. 1). Interpolation of the given points is guaranteed since the old points $\mathbf{p}_i^0 = \mathbf{p}_{2i}^1$ still belong to the finer approximation.

By iteratively applying this interpolatory refinement operation, a sequence of polygons $(\mathbf{P}_m)$ is generated with vertices becoming more and more dense and which satisfy the interpolation condition $\mathbf{p}_i^m = \mathbf{p}_{2i}^{m+1}$ for all i and m. This sequence may converge to a smooth limit $\mathbf{P}_\infty$.

Many authors have proposed different schemes by explicitly giving particular rules how to compute the new points $\mathbf{p}_{2i+1}^{m+1}$ as a function of the polygon $\mathbf{P}_m$ to be refined. In (Dubuc, 1986) a simple refinement scheme is proposed which uses four neighboring vertices to compute the position of a new point. The position is determined in terms of the unique cubic polynomial which uniformly interpolates these four points. The limiting curves generated by this scheme are smooth, i.e., they are differentiable with respect to an equidistant parametrisation.

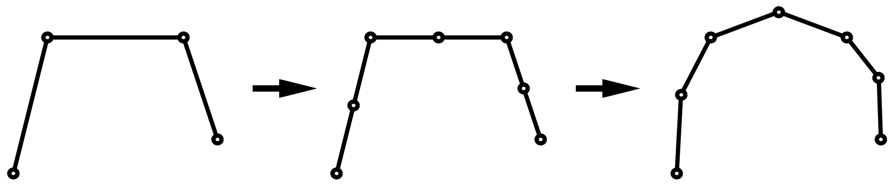

Figure 1: Interpolatory refinement

In (Dyn et al., 1987) this scheme is generalized by introducing an additional design or tension parameter. Replacing the interpolating cubic by interpolating polynomials of arbitrary degree leads to the *Lagrange-schemes* proposed in (Deslauriers & Dubuc, 1989). Raising the degree to $(2k+1)$, every new point depends on $(2k+2)$ old points of its vicinity. In (Kobbelt, 1995a) it is shown that at least for moderate k these schemes produce C^k-curves.

Appropriate formalisms have been developed in (Cavaretta et al., 1991), (Dyn & Levin, 1990), (Dyn, 1991) and elsewhere that allow an easy analysis of such *stationary schemes* which compute the new points by applying fixed banded convolution operators to the original polygon. In (Kobbelt, 1995b) simple criteria are given which can be applied to convolution schemes without any band limitation as well (cf. Theorem 2).

(Dyn et al., 1992) and (Le Méhauté & Utreras, 1994) propose non-linear refinement schemes which produce smooth interpolating (C^1-) curves and additionally preserve the convexity properties of the initial data. Both of them introduce constraints which locally define areas where the new points are restricted to lie in. Another possibility to define interpolatory refinement schemes is to dualize corner-cutting algorithms (Paluszny et al., 1994). This approach leads to more general necessary and sufficient convergence criteria.

In this paper we want to define interpolatory refinement schemes in a more systematic fashion. The major principle is the following: We are looking for refinement schemes for which, given a polygon $\mathbf{P}_m$, the refined polygon $\mathbf{P}_{m+1}$ is *as smooth as possible*. In order to be able to compare the "smoothness" of two polygons we define functionals $E(\mathbf{P}_{m+1})$ which measure the total amount of (discrete) strain energy of $\mathbf{P}_{m+1}$. The refinement operator then simply chooses the new points $\mathbf{p}_{2i+1}^{m+1}$ such that this functional becomes a minimum.

An important motivation for this approach is that in practice good approximations to the final interpolating curves should be achieved with little computational effort, i.e., maximum smoothness after a minimal number of refinement steps is wanted. In non-discrete curve design based, e.g., on splines, the concept of defining interpolating curves by the minimization of some energy functional (*fairing*) is very familiar (Meier & Nowacki, 1987), (Sapidis, 1994).

This basic idea of making a variational approach to the definition of refinement schemes can also be used for the definition of schemes which produce smooth surfaces by refining a given triangular or quarilateral net. However, due to the global dependence of the new points from the given net, the convergence analysis of such schemes strongly depends on the topology of the net to be refined and is still an open question. Numerical experiments with such schemes show that this approach is very promising. In this paper we will only address the analysis of univariate schemes.

2 Known results

Given an arbitrary (open/closed) polygon $\mathbf{P}_m = (\mathbf{p}_i^m)$, the *difference polygon* $\triangle^k \mathbf{P}_m$ denotes the polygon whose vertices are the vectors

$$\triangle^k \mathbf{p}_i^m := \sum_{j=0}^{k} \binom{k}{j} (-1)^{k+j} \, \mathbf{p}_{i+j}^m.$$

In (Kobbelt, 1995b) the following characterization of sequences of polygons $(\mathbf{P}_m)$ generated by the iterative application of an interpolatory refinement scheme is given:

Lemma 1 *Let $(\mathbf{P}_m)$ be a sequence of polygons. The scheme by which they are generated is an interpolatory refinement scheme (i.e., $\mathbf{p}_i^m = \mathbf{p}_{2i}^{m+1}$ for all i and m) if and only if for all $m, k \in I\!N$ the condition*

$$\triangle^k p_i^m = \sum_{j=0}^{k} \binom{k}{j} \triangle^k p_{2i+j}^{m+1}$$

holds for all indices i of the polygon $\triangle^k \mathbf{P}_m$.

Also in (Kobbelt, 1995b), the following sufficient convergence criterion is proven which we will use in the convergence analysis in the next sections.

Theorem 2 *Let $(\mathbf{P}_m)$ be a sequence of polygons generated by the iterative application of an arbitrary interpolatory refinement scheme. If*

$$\sum_{m=0}^{\infty} \|2^{km} \triangle^{k+l} \mathbf{P}_m\|_\infty < \infty$$

for some $l \in I\!N$ then the sequence $(\mathbf{P}_m)$ uniformly converges to a k-times continuously differentiable curve $\mathbf{P}_\infty$.

This theorem holds for all kinds of interpolatory schemes on open and closed polygons. However, in this paper we will only apply it to linear schemes whose support is global.

3 A variational approach to interpolatory refinement

In this and the next two sections we focus on the refinement of *closed* polygons, since this simplifies the description of the refinement schemes. Open polygons will be considered in Section 6.

Let $\mathbf{P}_m = (\mathbf{p}_0^m, \ldots, \mathbf{p}_{n-1}^m)$ be a given polygon. We want $\mathbf{P}_{m+1} = (\mathbf{p}_0^{m+1}, \ldots, \mathbf{p}_{2n-1}^{m+1})$ to be the smoothest polygon for which the interpolation condition $\mathbf{p}_{2i}^{m+1} = \mathbf{p}_i^m$ holds. Since the roughness at some vertex $\mathbf{p}_i^{m+1}$ is a local property we measure it by a an operator

$$K(\mathbf{p}_i^{m+1}) := \sum_{j=0}^{k} \alpha_j \, \mathbf{p}_{i+j-r}^{m+1}.$$

The coefficients α_j in this definition can be an arbitrary finite sequence of real numbers. The indices of the vertices $\mathbf{p}_i^{m+1}$ are taken *modulo* $2n$ according to the topological structure of the closed polygon $\mathbf{P}_{m+1}$. To achieve full generality we introduce the shift r such that $K(\mathbf{p}_i^{m+1})$ depends on $\mathbf{p}_{i-r}^{m+1}, \ldots, \mathbf{p}_{i+k-r}^{m+1}$. Every discrete measure of roughness K is associated with a characteristic polynomial

$$\alpha(z) = \sum_{j=0}^{k} \alpha_j \, z^j.$$

Our goal is to minimize the total strain energy over the whole polygon $\mathbf{P}_{m+1}$. Hence we define

$$E(\mathbf{P}_{m+1}) := \sum_{i=0}^{2n-1} K(\mathbf{p}_i^{m+1})^2 \tag{1}$$

to be the energy functional which should become minimal. Since the points $\mathbf{p}_{2i}^{m+1}$ of $\mathbf{P}_{m+1}$ with even indices are fixed due to the interpolation condition, the points $\mathbf{p}_{2i+1}^{m+1}$ with odd indices are the only free parameters of this optimization problem. The unique minimum of the quadratic functional is attained at the common root of all partial derivatives:

$$\frac{\partial}{\partial \mathbf{p}_{2l+1}^{m+1}} E(\mathbf{P}_{m+1}) = \sum_{i=0}^{k} \frac{\partial}{\partial \mathbf{p}_{2l+1}^{m+1}} K(\mathbf{p}_{2l+1+r-i}^{m+1})^2$$

$$= 2 \sum_{i=0}^{k} \alpha_i \sum_{j=0}^{k} \alpha_j \, \mathbf{p}_{2l+1-i+j}^{m+1} \qquad (2)$$

$$= 2 \sum_{i=-k}^{k} \beta_i \, \mathbf{p}_{2l+1+i}^{m+1}$$

with the coefficients

$$\beta_{-i} = \beta_i = \sum_{j=0}^{k-i} \alpha_j \, \alpha_{j+i}, \qquad i = 0, \ldots, k. \qquad (3)$$

Hence, the strain energy $E(\mathbf{P}_{m+1})$ becomes minimal if the new points $\mathbf{p}_{2i+1}^{m+1}$ are the solution of the linear system

$$\begin{pmatrix} \beta_0 & \beta_2 & \beta_4 & \ldots & \beta_2 \\ \beta_2 & \beta_0 & \beta_2 & \ldots & \beta_4 \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \begin{pmatrix} \mathbf{p}_1^{m+1} \\ \mathbf{p}_3^{m+1} \\ \vdots \\ \mathbf{p}_{2n-1}^{m+1} \end{pmatrix} = \begin{pmatrix} -\beta_1 & -\beta_1 & -\beta_3 & \ldots & -\beta_3 \\ -\beta_3 & -\beta_1 & -\beta_1 & \ldots & -\beta_5 \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \begin{pmatrix} \mathbf{p}_0^{m} \\ \mathbf{p}_1^{m} \\ \vdots \\ \mathbf{p}_{n-1}^{m} \end{pmatrix} \qquad (4)$$

which follows from (2) by separation of the fixed points $\mathbf{p}_{2i}^{m+1} = \mathbf{p}_i^m$ from the variables. Here, both matrices are circulant and (almost) symmetric. A consequence of this symmetry is that the new points do not depend on the orientation by which the vertices are numbered (left to right or vice versa).

To emphasize the analogy between curve fairing and interpolatory refinement by variational methods, we call the equation

$$\sum_{i=-k}^{k} \beta_i \, \mathbf{p}_{2l+1+i}^{m+1} = 0, \qquad l = 0, \ldots, n-1 \qquad (5)$$

the *Euler-Lagrange-equation*.

Theorem 3 *The minimization of $E(\mathbf{P}_{m+1})$ has a well-defined solution if and only if the characteristic polynomial $\alpha(z)$ for the local measure K has no diametric roots $z = \pm\omega$ on the unit circle with $Arg(\omega) \in \pi\, \mathbb{N}/n$.*

Proof Let n be the number of points in the unrefined polygon $\mathbf{P}_m$. The minimization is uniquely solvable if and only if the matrix $\mathcal{B} = \mathrm{Circ}[\beta_0, \beta_2, \ldots]$ on the left side of (4) is regular. We define the $(2n \times 2n)$ matrix $\mathcal{C} = \mathrm{Circ}[\alpha_0, \ldots, \alpha_k, 0, \ldots]$ using the coefficients

α_j of K. The matrix $\mathcal{A}$ is then obtained from $\mathcal{C}$ by deleting every second column. Since we have

$$\mathcal{B} \;=\; \mathcal{A}^T \mathcal{A}$$

it follows that $\mathcal{B}$ is regular if and only if $\mathcal{A}$ has full rank n (Golub & Van Loan, 1989).

The eigen values of the circulant matrix $\mathcal{C}$ can be computed by applying the discrete Fourier transform:

$$\lambda_i \;=\; \sum_{j=0}^{k} \alpha_j \, \omega_{2n}^{ij} \;=\; \alpha(\omega_{2n}^{i}), \qquad i = 0, \ldots, 2n-1,$$

where ω_{2n} is a $2n$-th root of unity. Hence, the eigen values λ_i of $\mathcal{C}$ follow from sampling the characteristic polynomial $\alpha(z)$ equidistantly over the unit circle. The corresponding eigen vectors are $\mathbf{v}_i = (\omega_{2n}^{ij})_{j=0}^{2n-1}$.

The existence of diametric roots ω_{2n}^{i} and $-\omega_{2n}^{i} = \omega_{2n}^{i+n}$ implies that both eigen values λ_i and λ_{i+n} vanish. The corresponding eigen vectors are $\mathbf{v}_i = (\omega_{2n}^{ij})_{j=0}^{2n-1}$ and $\mathbf{v}_{i+n} = ((-1)^j \, \omega_{2n}^{ij})_{j=0}^{2n-1}$ and the subspace spanned by these two vectors also contains the vector $\mathbf{v}_i + \mathbf{v}_{i+n}$ for which every second component is zero. The non-vanishing components of this vector constitute a linear combination of the columns of $\mathcal{A}$ that obtains the zero-vector. Thus $\mathcal{A}$ must be rank deficient.

On the other hand: suppose $\mathcal{A}$ is rank deficient. Then there is a non-trivial linear combination of the columns of $\mathcal{A}$ representing the zero-vector. The coefficients of this combination can obviously be expanded to a vector of the kernel of $\mathcal{C}$ by inserting alternating zero components. This kernel vector lies in the n dimensional subspace spanned by the vectors $\mathbf{v}_i + \mathbf{v}_{i+n}$ for $i = 0, \ldots, n-1$. Since the $\mathbf{v}_i$ are linearly independent, for at least one index i the condition $\lambda_i = \lambda_{i+n} = 0$ must hold and this is equivalent to the existence of diametric roots of $\alpha(z)$ on the unit circle. $\square$

Remark The set $\pi \, \mathbb{N}/2^m$ becomes dense in $\mathbb{R}$ for increasing refinement depth $m \to \infty$. Since we are interested in the smoothness properties of the limiting curve $\mathbf{P}_\infty$, we should drop the restriction that the diametric roots have to have $\mathrm{Arg}(\omega) \in \pi \, \mathbb{N}/n$. For *stability* reasons we require $\alpha(z)$ to have no diametric roots on the unit circle at all.

The optimization by which the new points are determined is a geometric process. In order to obtain meaningful schemes, we have to introduce more restrictions on the energy functionals E or on the measures of roughness K.

For the expression $K^2(\mathbf{p}_i)$ to be valid, K has to be vector valued, i.e., the sum of the coefficients α_j has to be zero. This is equivalent to $\alpha(1) = 0$. Since

$$\sum_{i=-k}^{k} \beta_i = \sum_{i=0}^{k} \sum_{j=0}^{k} \alpha_i \, \alpha_j = \Big(\sum_{j=0}^{k} \alpha_j \Big)^2$$

the sum of the coefficients β_i also vanishes in this case and *affine invariance* of the (linear) scheme is guaranteed because constant functions are reproduced.

4 Implicit refinement schemes

In the last section we showed that the minimization of a quadratic energy functional (1) leads to the conditions (5) which determine the solution. Dropping the variational background, we can more generally prescribe arbitrary real coefficients $\beta_{-k}, \ldots, \beta_k$ (with $\beta_{-i} = \beta_i$ to establish symmetry and $\sum \beta_i = 0$ for affine invariance) and define an interpolatory refinement scheme which chooses the new points $\mathbf{p}_{2i+1}^{m+1}$ of the refined polygon $\mathbf{P}_{m+1}$ such that the homogeneous constraints

$$\sum_{i=-k}^{k} \beta_i \, \mathbf{p}_{2l+1+i}^{m+1} \;=\; 0, \qquad l = 0, \ldots, n-1 \tag{6}$$

are satisfied. We call these schemes: *implicit refinement schemes* to emphasize the important difference to other refinement schemes where usually the new points are computed by one or two *explicitly* given rules (cf. the term *implicit curves* for curves represented by $f(x, y) = 0$). The stationary refinement schemes are a special case of the implicit schemes where $\beta_{2j} = \delta_{j,0}$. In general, the implicit schemes are non-stationary since the resulting weight coefficients by which the new points $\mathbf{p}_{2i+1}^{m+1}$ are computed depend on the number of vertices in $\mathbf{P}_m$.

In (Kobbelt, 1995b) a general technique is presented which allows to analyse the smoothness properties of the limiting curve generated by a given implicit refinement scheme.

The next theorem reveals that the class of implicit refinement schemes is not essentially larger than the class of variational schemes.

Theorem 4 *Let $\beta_{-k}, \ldots, \beta_k$ be an arbitrary symmetric set of real coefficients ($\beta_{-i} = \beta_i$). Then there always exists a (potentially complex valued) local roughness measure K such that (6) is the Euler-Lagrange-equation corresponding to the minimization of the energy functional (1).*

Proof We define the characteristic Laurent-polynomial of (6):

$$\beta(z) \;:=\; \sum_{i=-k}^{k} \beta_i \, z^i.$$

From (3) it is easy to see that a characteristic polynomial $\alpha(z)$ of some local measure K is related to the Laurent-polynomial of the corresponding Euler-Lagrange-equation by

$$\beta(z) \;=\; \alpha(z) \, \alpha(z^{-1}).$$

Hence we have to show that a factorization of this form is always possible. Without loss of generality we assume $\beta_{-k} = \beta_k \neq 0$. Thus $\beta(z)$ can be written as

$$\beta(z) = \beta_k \, z^{-k} \prod_{i=1}^{2k} (z - z_i)$$

where z_i are the complex roots of $\beta(z)$ with all $z_i \neq 0$ since

$$\prod_{i=1}^{2k} z_i = 1.$$

From the symmetry of the coefficients β_i it follows that

$$\beta(z) \;=\; \beta(z^{-1}) \;=\; \beta_k \, z^k \prod_{i=1}^{2k} (z^{-1} - z_i).$$

Thus for every root z_i of $\beta(z)$ there exists another root z_j with the same multiplicity such that $z_i \, z_j = 1$. Since the total number of roots is even and every $z_i \neq 1$ has a "partner" of equal multiplicity, the root $z = 1$, if it occurs, must have even multiplicity. Hence $\beta(z)$ can be factorized into

$$\begin{aligned}
\beta(z) &= \beta_k \, z^{-k} \prod_{i=1}^{k} (z - z_i) \prod_{i=1}^{k} (z - z_i^{-1}) \\
&= \beta_k \prod_{i=1}^{k} (z - z_i) \prod_{i=1}^{k} (1 - z^{-1} \, z_i^{-1}) \\
&= (-1)^k \, \beta_k \prod_{i=1}^{k} z_i^{-1} \prod_{i=1}^{k} (z - z_i) \prod_{i=1}^{k} (z^{-1} - z_i).
\end{aligned}$$

The multiplication of the coefficients β_i by a common factor does not affect the solution of (6) nor changes the roots z_i of $\beta(z)$. Therefore we can use

$$\alpha(z) = \prod_{i=1}^{k} (z - z_i).$$

as the characteristic polynomial of the local roughness measure K. □

Obviously the factorization $\beta(z) = \alpha(z) \, \alpha(z^{-1})$ is not unique.

Remark We do not consider implicit refinement schemes with complex coefficients β_i since then (6) in general has no real solutions.

Example To illustrate the statement of the last theorem we look at the 4-point scheme of (Dubuc, 1986). This is a stationary refinement scheme where the new points $\mathbf{p}_{2i+1}^{m+1}$ are computed by the rule

$$\mathbf{p}_{2i+1}^{m+1} = \frac{9}{16}\,(\mathbf{p}_i^m + \mathbf{p}_{i+1}^m) - \frac{1}{16}\,(\mathbf{p}_{i-1}^m + \mathbf{p}_{i+2}^m).$$

The scheme can be written in implicit form (6) with $k = 3$ and $\beta_{\pm 3} = 1$, $\beta_{\pm 2} = 0$, $\beta_{\pm 1} = -9$, $\beta_0 = 16$ since the common factor $\frac{1}{16}$ is not relevant. The roots of $\beta(z)$ are $z_1 = \ldots = z_4 = 1$ and $z_{5,6} = -2 \pm \sqrt{3}$. From the construction of the last proof we obtain

$$\alpha(z) = (2+\sqrt{3}) - (3+\sqrt{12})\,z + \sqrt{3}\,z^2 + z^3$$

as one possible solution. Hence, the quadratic strain energy which is minimized by the 4-point scheme is based on the local roughness estimate

$$K(\mathbf{p}_i) = (2+\sqrt{3})\,\mathbf{p}_i - (3+\sqrt{12})\,\mathbf{p}_{i+1} + \sqrt{3}\,\mathbf{p}_{i+2} + \mathbf{p}_{i+3}.$$

5 Minimization of differences

Theorem 2 asserts that a fast contraction rate of some higher differences is sufficient for the convergence of a sequence of polygons to a (k times) continuously differentiable limiting curve. Thus it is natural to look for refinement schemes with a maximum contraction of differences. This obviously is an application of the variational approach. For the quadratic energy functional we make the ansatz

$$E_k(\mathbf{P}_{m+1}) := \sum_{i=0}^{2n-1} \|\triangle^k \mathbf{p}_i^{m+1}\|^2. \tag{7}$$

The partial derivatives take a very simple form in this case

$$\begin{aligned}
\frac{\partial}{\partial\,\mathbf{p}_{2l+1}^{m+1}}\,E_k(\mathbf{P}_{m+1}) &= \sum_{i=0}^{k} \frac{\partial}{\partial\,\mathbf{p}_{2l+1}^{m+1}}\,\|\triangle^k \mathbf{p}_{2l+1-i}^{m+1}\|^2 \\
&= 2\sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} \triangle^k \mathbf{p}_{2l+1-i}^{m+1} \\
&= 2\,(-1)^k\,\triangle^{2k} \mathbf{p}_{2l+1-k}^{m+1}.
\end{aligned}$$

and the corresponding Euler-Lagrange-equation is

$$\triangle^{2k} \mathbf{p}_{2l+1-k}^{m+1} = 0, \qquad l = 0, \ldots, n-1 \tag{8}$$

where, again, the indices of the $\mathbf{p}_i^{m+1}$ are taken *modulo* $2n$. The characteristic polynomial of the underlying roughness measure K is $\alpha(z) = (z-1)^k$ and thus solvability and affine invariance of the refinement scheme are guaranteed. The solution of (8) only requires the inversion of a banded circulant matrix with bandwidth $2\left\lfloor\frac{k}{2}\right\rfloor + 1$.

Theorem 5 *The refinement scheme based on the minimization of E_k in (7) produces at least C^k-curves.*

Proof The proof will be done in two steps. First we show that $\mathbf{P}_\infty \in C^{k-1}$ and then $\mathbf{P}_\infty \in C^k$. We will analyse the contraction of the $2k$-th differences and then apply Theorem 2.

Lemma 1 states the following relation between components of successive difference polygons generated by an arbitrary interpolatory refinement scheme

$$\triangle^{2k}\mathbf{p}_i^m = \sum_{j=0}^{2k} \binom{2k}{j} \triangle^{2k}\mathbf{p}_{2i+j}^{m+1}.$$

From the Euler-Lagrange-equation (8) it follows that every second component of the difference polygon $\triangle^{2k}\,\mathbf{P}_{m+1}$ vanishes. Hence we have for even k

$$\triangle^{2k}\mathbf{p}_i^m = \sum_{j=0}^{k} \binom{2k}{2j} \triangle^{2k}\mathbf{p}_{2i+2j}^{m+1}$$

and for odd k

$$\triangle^{2k}\mathbf{p}_i^m = \sum_{j=1}^{k} \binom{2k}{2j-1} \triangle^{2k}\mathbf{p}_{2i+2j-1}^{m+1}.$$

Let $\mathbf{P}_m = (\mathbf{p}_0^m, \ldots, \mathbf{p}_{n-1}^m)$ be the polygon to be refined. The non-vanishing differences $\triangle^{2k}\mathbf{p}_{2i+k}^{m+1}$ are computed from $\triangle^{2k}\,\mathbf{P}_m$ by solving

$$\mathcal{B}\,(\triangle^{2k}\mathbf{p}_{2i+k}^{m+1})_{i=0}^{n-1} \;=\; (\triangle^{2k}\mathbf{p}_i^m)_{i=0}^{n-1}$$

with

$$\mathcal{B} = \mathrm{Circ}\left[\binom{2k}{k}, \binom{2k}{k+2}, \ldots, \binom{2k}{k-4}, \binom{2k}{k-2}\right]. \tag{9}$$

Notice that the binomial coefficients vanish, if the lower number does not lie in the set $[0, 2k] \cap \mathbb{N}$. To estimate the rate of contraction for the $2k$-th differences we need to know the smallest eigen value of $\mathcal{B}$ (or the largest eigen value of $\mathcal{B}^{-1}$). The eigen values λ_i of $\mathcal{B}$ can be computed by applying the Fourier transform

$$\lambda_i = \sum_{j=-\lfloor n/2 \rfloor}^{\lceil n/2 \rceil -1} \binom{2k}{k+2j} \omega_n^{ij}, \qquad i = 0, \ldots, n-1.$$

Here ω_n denotes a n-th root of unity. Due to the symmetry of $\mathcal{B}$ all λ_i are real. The spectrum of $\mathcal{B}$ can be bounded from below by the minimum of the polynomial

$$\lambda(\omega) = \sum_{j=-\lfloor n/2 \rfloor}^{\lceil n/2 \rceil -1} \binom{2k}{k+2j} \omega^{j}. \tag{10}$$

on the unit circle $|\omega| = 1$. Let $z^4 = \omega$

$$\begin{aligned}
\lambda(z^4) &= \frac{1}{2}\left(\sum_{i=0}^{2k} \binom{2k}{i} z^{2i} + (-1)^k \sum_{i=0}^{2k} \binom{2k}{i} (-z^2)^i \right) z^{-2k} \\
&= \frac{1}{2}\left((1+z^2)^{2k} + (-1)^k (1-z^2)^{2k} \right) z^{-2k} \\
&= \frac{1}{2}\left((z^{-1}+z)^{2k} + (-1)^k (z^{-1}-z)^{2k} \right) \\
&= 2^{2k-1}\left(\cos(x)^{2k} + \sin(x)^{2k}\right) \qquad > 0,
\end{aligned}$$

where $z = cos(x) + \hat{\imath}\,\sin(x)$. The last term becomes minimal for $|\sin(x)| = |\cos(x)|$ or $z^4 = -1$. Thus the smallest eigen value of $\mathcal{B}$ is

$$\lambda_{\lfloor n/2 \rfloor} \;\geq\; \lambda(-1) \;=\; 2^k.$$

Having the upper bound $\rho = 2^{-k}$ for the spectral radius of $\mathcal{B}^{-1}$ we obtain

$$\sum_{i=0}^{2n-1} \|\triangle^{2k} \mathbf{p}_i^{m+1}\|^2 \;=\; \sum_{i=0}^{n-1} \|\triangle^{2k} \mathbf{p}_{2i+k}^{m+1}\|^2 \;\leq\; \rho^2 \sum_{i=0}^{n-1} \|\triangle^{2k} \mathbf{p}_i^{m}\|^2$$

and

$$\|\triangle^{2k}\,\mathbf{P}_{m+1}\|_\infty \;\leq\; \sqrt{\sum_{i=0}^{2n-1} \|\triangle^{2k} \mathbf{p}_i^{m+1}\|^2} \;\leq\; \sigma\,\rho^{m+1}$$

with $\sigma \in \mathbb{R}$ only depending on $\mathbf{P}_0$. Hence

$$\|2^{(k-1)\,m}\,\triangle^{2k}\,\mathbf{P}_m\|_\infty = O(2^{-m})$$

and the sufficient condition of Theorem 2 is satisfied for $l = k + 1$.

So far the contraction analysis is done by looking at one single refinement step applied to an arbitrary polygon $\mathbf{P}_m$. However, from the second step on, the polygon $\mathbf{P}_m$ to be refined is no longer arbitrary. This can be exploited to improve the lower bound on the differentiability of $\mathbf{P}_\infty$.

The eigen values of the matrix $\mathcal{B}$ in (9) are computed by sampling the polynomial (10) equidistantly on the unit circle $|\omega| = 1$. The eigen vector $\mathbf{v}_i = (\omega_n^{ij})_{j=0}^{n-1}$ corresponds to the eigen value $\lambda_i = \lambda(\omega_n^i)$, where, again, ω_n denotes a n-th root of unity.

For $m \geq 1$ the polygon $\mathbf{P}_m$ itself is the result of a refinement operation. Thus the number of vertices n of $\mathbf{P}_m$ is even and every second component $\triangle^{2k}\mathbf{p}^m_{2i+1-k}$ of its difference polygon $\triangle^{2k}\,\mathbf{P}_m$ is zero. Consequently $\triangle^{2k}\,\mathbf{P}_m$ lies in the $\frac{n}{2}$-dimensional subspace which is spanned by the vectors $\mathbf{w}_i = \mathbf{v}_i + (-1)^k\,\mathbf{v}_{i+n/2}$ and there exist coefficients γ_i such that

$$\begin{aligned}
\mathcal{B}^{-1}\,\triangle^{2k}\,\mathbf{P}_m &= \sum_{i=0}^{n/2-1} \gamma_i\;\mathcal{B}^{-1}\,\mathbf{w}_i \\
&= \sum_{i=0}^{n/2-1} \gamma_i \left(\frac{1}{\lambda_i}\,\mathbf{v}_i + (-1)^k\,\frac{1}{\lambda_{i+n/2}}\,\mathbf{v}_{i+n/2}\right).
\end{aligned}$$

Since the $\mathbf{v}_i$ form an orthogonal basis and $\|\mathbf{v}_i\|_2 = \sqrt{n}$, we obtain

$$\|\triangle^{2k}\,\mathbf{P}_m\|_2^2 \;=\; n \sum_{i=0}^{n/2-1} 2\,|\gamma_i|^2$$

and

$$\|\mathcal{B}^{-1}\,\triangle^{2k}\,\mathbf{P}_m\|_2^2 \;=\; n \sum_{i=0}^{n/2-1} (\lambda_i^{-2} + \lambda_{i+n/2}^{-2})\,|\gamma_i|^2$$

by the theorem of Pythagoras. Thus we have to bound the maximum value of

$$\sqrt{\frac{\lambda(\omega)^{-2} + \lambda(-\omega)^{-2}}{2}}, \qquad |\omega| = 1.$$

Since $\omega = -1$ is the only point on the unit circle where $\lambda(\omega)$ attains its minimum $\lambda(-1) = 2^k$ we have

$$\mu(\omega) := \lambda(\omega)^{-2} + \lambda(-\omega)^{-2} < 2^{1-2k}.$$

Furthermore, since $\mu(\omega)$ is continuous, it actually attains its maximum on the compact unit circle. Thus there exists a $q < 2$ such that $\mu(\omega) \leq q\,2^{-2k}$. From this follows a contraction rate of

$$\|\triangle^{2k}\mathbf{P}_{m+1}\|_2 \;\leq\; 2^{-k}\sqrt{\frac{q}{2}}\;\|\triangle^{2k}\mathbf{P}_m\|_2$$

and consequently

$$\|2^{mk}\triangle^{2k}\mathbf{P}_m\|_\infty \;=\; O\Big(\sqrt{\frac{q}{2}}^{\,m}\Big)$$

which is sufficient for $\mathbf{P}_\infty$ to be C^k. $\square$

In order to prove even higher regularities of the limiting curve one has to combine more refinement steps. In (Kobbelt, 1995b) a simple technique is presented that allows to do the convergence analysis of such multi-step schemes numerically. Table 1 shows some results where r denotes the number of step that have to be combined in order to obtain these differentiabilities.

In analogy to the non-discrete case where the minimization of the integral over the squared k-th derivative has piecewise polynomial C^{2k-2} solutions (B-splines), it is very likely that the limiting curves generated by iterative minimization of E_k are actually in C^{2k-2} too. The results given in Table 1 can be improved by combining more than r steps. For $k = 2, 3$, however, sufficiently many steps have already been combined to verify $\mathbf{P}_\infty \in C^{2k-2}$.

k	r	diff'ty	k	r	diff'ty
2	2	C^2	7	6	C^{10}
3	11	C^4	8	4	C^{11}
4	2	C^5	9	6	C^{13}
5	7	C^7	10	4	C^{14}
6	3	C^8	11	6	C^{16}

Table 1: Lower bounds on the differentiability of $\mathbf{P}_\infty$ generated by iterative minimization of $E_k(\mathbf{P}_m)$.

For illustration and to compare the quality of the curves generated by these schemes, some examples are given in Fig. 2. The curves result from applying different schemes to the initial data $\mathbf{P}_0 = (\ldots, 0, 1, 0, \ldots)$. We only show the middle part of one periodic interval of $\mathbf{P}_\infty$. As expected, the decay of the function becomes slower as the smoothness increases.

Remark Considering Theorem 2 it would be more appropriate to minimize the maximum difference $\|\triangle^k\mathbf{P}_m\|_\infty$ instead of $\|\triangle^k\mathbf{P}_m\|_2$. However, this leads to non-linear refinement schemes which are both, hard to compute and difficult to analyse. Moreover, in (Kobbelt, 1995a) it is shown that a contraction rate of $\|\triangle^{2k}\mathbf{P}_m\|_\infty = O(2^{-mk})$ implies

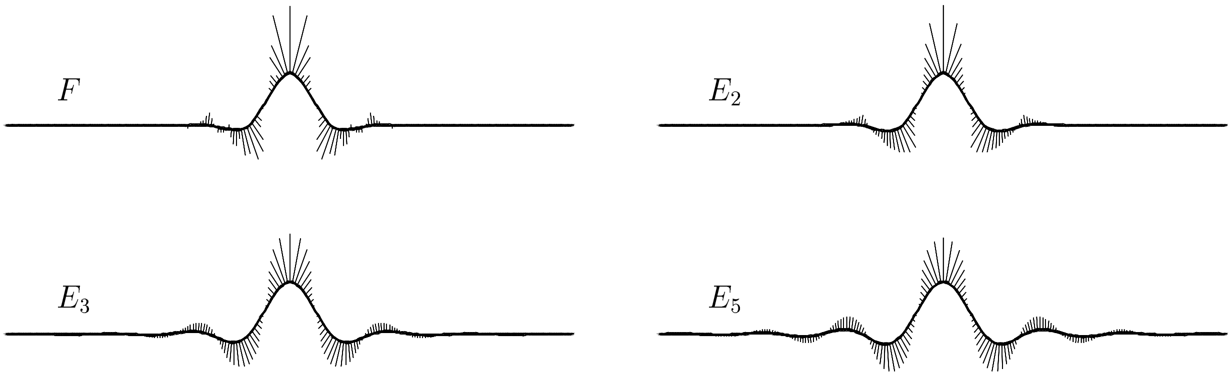

Figure 2: Discrete curvature plots of finite approximations to the curves generated by the four-point scheme F ($\mathbf{P}_\infty \in C^1$) and the iterative minimization of E_2 ($\mathbf{P}_\infty \in C^2$), E_3 ($\mathbf{P}_\infty \in C^4$) and E_5 ($\mathbf{P}_\infty \in C^7$).

$\|\triangle^k \mathbf{P}_m\|_\infty = O(2^{-m(k-\varepsilon)})$ for every $\varepsilon > 0$. It is further shown that $\|\triangle^k \mathbf{P}_m\|_\infty = O(2^{-mk})$ is the theoretical fastest contraction which can be achieved by interpolatory refinement schemes. Hence, the minimization of $\|\triangle^k \mathbf{P}_m\|_\infty$ cannot improve the asymptotic behavior of the contraction.

6 Interpolatory refinement of open polygons

The convergence analysis of variational schemes in the case of open finite polygons is much more difficult than it is in the case of closed polygons. The problems arise at both ends of the polygons $\mathbf{P}_m$ where the regular topological structure is disturbed. Therefore, we can no longer describe the refinement operation in terms of Toeplitz matrices but we have to use matrices which are Toeplitz matrices almost everywhere except for a finite number of rows, i.e., except for the first and the last few rows.

However, one can show that in a middle region of the polygon to be refined the smoothing properties of an implicit refinement scheme applied to an open polygon do not differ very much from the same scheme applied to a closed polygon. This is due to the fact that in both cases the influence of the old points $\mathbf{p}_i^m$ on a new point $\mathbf{p}_{2j+1}^{m+1}$ decrease exponentially with increasing topological distance $|i-j|$ for all asymptotically stable schemes (Kobbelt, 1995a).

For the refinement schemes which iteratively minimize forward differences, we can at least prove the following.

Theorem 6 *The interpolatory refinement of open polygons by iteratively minimizing the $2k$-th differences, generates at least C^{k-1}-curves.*

Proof The basic idea of this proof is to construct a reference refinement scheme which achieves a certain rate of contraction for the $2k$-th differences. Then it is obvious that the *minimization* of the $2k$-th differences achieves at least the same rate and we can apply Theorem 2.

Let $\mathbf{P}_m = (\mathbf{p}_0^m, \ldots, \mathbf{p}_n^m)$ be a given open polygon, where m is chosen large enough such that $n \geq 2k$. We extend the difference polygon $\triangle^{2k}\,\mathbf{P}_m = (\triangle^{2k}\mathbf{p}_0^m, \ldots, \triangle^{2k}\mathbf{p}_{n-2k}^m)$ to a biinfinite polygon $\triangle^{2k}\,\mathbf{Q}_m$ by adding zero components on both sides.

We refine by first applying the biinfinite Toeplitz operator $\mathcal{B}^{-1}$ with $\mathcal{B} = (b_{ij})$ and

$$b_{ij} \;=\; \binom{2k}{k+2\,(j-i)}, \qquad -\infty \;\leq\; i,j \;\leq\; \infty$$

and then introduce alternating zero-components by applying $\mathcal{C} = (\delta_{i,2j+k})$. Finally we cut some middle region $\triangle^{2k}\,\widetilde{\mathbf{P}}_{m+1} := (\triangle^{2k}\mathbf{q}_0^{m+1}, \ldots, \triangle^{2k}\mathbf{q}_{2n-2k}^{m+1})$ from the resulting polygon $\triangle^{2k}\,\mathbf{Q}_{m+1} = (\triangle^{2k}\mathbf{q}_i^{m+1}) = \mathcal{C}\,\mathcal{B}^{-1}\,\triangle^{2k}\,\mathbf{Q}_m$. The components of $\triangle^{2k}\,\mathbf{P}_m$ and $\triangle^{2k}\,\widetilde{\mathbf{P}}_{m+1}$ are related by

$$\begin{aligned}
\triangle^{2k}\mathbf{p}_i^m \;&=\; \sum_{j=-\infty}^{\infty} \binom{2k}{k+2\,(j-i)}\,\triangle^{2k}\mathbf{q}_{2j+k}^{m+1} \\
&=\; \sum_{j=-\infty}^{\infty} \binom{2k}{k-2i+j}\,\triangle^{2k}\mathbf{q}_{j+k}^{m+1} \\
&=\; \sum_{j=0}^{2k} \binom{2k}{j}\,\triangle^{2k}\mathbf{q}_{2i+j}^{m+1}.
\end{aligned}$$

Hence, $\triangle^{2k}\,\widetilde{\mathbf{P}}_{m+1}$ is the difference polygon corresponding to a possible interpolatory refinement $\widetilde{\mathbf{P}}_{m+1}$ of $\mathbf{P}_m$ (cf. Lemma 1).

The construction we used can be considered as the refinement of $\mathbf{Q}_m$ by minimizing the k-th differences (cf. proof of Theorem 5). The spectrum of $\mathcal{B}$ is the range of the polynomial

$$\lambda(\omega) \;=\; \sum_{j=-\lfloor\frac{k}{2}\rfloor}^{\lfloor\frac{k}{2}\rfloor} \binom{2k}{k+2j}\,\omega^j$$

over the unit circle (Widom, 1965). Hence, from the proof of Theorem 5 we know that the spectral radius of $\mathcal{B}^{-1}$ is bounded by $\rho = 2^{-k}$. We have

$$\sum_{i=0}^{2n-2k} \|\triangle^{2k}\mathbf{q}_i^{m+1}\|^2 \;\leq\; \sum_{i=-\infty}^{\infty} \|\triangle^{2k}\mathbf{q}_{2i+k}^{m+1}\|^2 \;\leq\; \rho^2 \sum_{i=0}^{n-2k} \|\triangle^{2k}\mathbf{p}_i^m\|^2,$$

where we exploit the fact that the $\triangle^{2k}\mathbf{q}_{2i+1+k}^{m+1}$ for all i and the $\triangle^{2k}\mathbf{p}_i^m$ for $i < 0$ or $i > n-2k$ vanish.

Now consider the difference polygon $\triangle^{2k}\mathbf{P}_{m+1}$ of $\mathbf{P}_{m+1}$ which is obtained from $\mathbf{P}_m$ by the refinement scheme that minimizes the $2k$-th differences. Since the contraction for this scheme must be at least as fast as for the reference scheme above, we obtain

$$\sum_{i=0}^{2n-2k} \|\triangle^{2k}\mathbf{p}_i^{m+1}\|^2 \ \le\ \rho^2 \sum_{i=0}^{n-2k} \|\triangle^{2k}\mathbf{p}_i^{m}\|^2.$$

The standard estimation $\|\cdot\|_\infty \le \|\cdot\|_2$ yields

$$\|\triangle^{2k}\mathbf{P}_m\|_\infty \ =\ O(2^{-mk})$$

which together with Theorem 2 concludes the proof. $\square$

The statement of this theorem only gives a lower bound for the differentiability of the limiting curve $\mathbf{P}_\infty$. However, the author conjects that the differentiabilities agree in the open and closed polygon case. For special cases we can prove better results.

Theorem 7 *The interpolatory refinement of open polygons by iteratively minimizing the second differences, generates at least C^2-curves.*

Proof Let $\mathbf{P}_m = (\mathbf{p}_0^m, \ldots, \mathbf{p}_n^m)$ be a given open polygon. The Euler-Lagrange-conditions for the minimization of the second differences at the inner vertices, i.e., the partial derivatives of the energy functional

$$E_2(\mathbf{P}_{m+1}) \ =\ \sum_{i=0}^{2n-2} \|\triangle^2\mathbf{p}_i^{m+1}\|^2$$

with respect to the free points $\mathbf{p}_{2l+1}^{m+1}$ with $l = 1, \ldots, n-2$ are

$$\triangle^4\mathbf{p}_{2l-1}^{m+1} = 0, \qquad l = 1, \ldots, n-2. \tag{11}$$

The roots of the partial derivative with respect to the first free point $\mathbf{p}_1^{m+1}$ yields the auxiliary condition

$$2\,\triangle^2\mathbf{p}_0^{m+1} \ =\ \triangle^2\mathbf{p}_1^{m+1}.$$

Putting this together with (11) for $l = 1$ and Lemma 1 for $k = 2$ yields

$$6\triangle^4\mathbf{p}_0^{m+1} + \triangle^4\mathbf{p}_2^{m+1} = \triangle^2\mathbf{p}_1^m - 2\,\triangle^2\mathbf{p}_0^m$$

where the right side vanishes for $m \ge 1$. A similar condition is obtained at the opposite end. Hence for $m \ge 1$ the non-vanishing components of $\triangle^4\mathbf{P}_{m+1}$ are the solution of the equation system

$$\begin{pmatrix} 6 & 1 & 0 & \dots & 0 \\ 1 & 6 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 6 & 1 \\ 0 & \dots & 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} \triangle^4 \mathbf{p}_0^{m+1} \\ \triangle^4 \mathbf{p}_2^{m+1} \\ \vdots \\ \triangle^4 \mathbf{p}_{2n-6}^{m+1} \\ \triangle^4 \mathbf{p}_{2n-4}^{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \triangle^4 \mathbf{p}_0^m \\ \vdots \\ \triangle^4 \mathbf{p}_{n-4}^m \\ 0 \end{pmatrix}, \tag{12}$$

where we apply Lemma 1 as in the proof of Theorem 5. From the Theorem of Gerschgorin it follows that the smallest eigen value of this matrix is ≥ 4. The coefficients of the inverse matrix $\mathcal{A}^{-1} = (\alpha_{ij})_{i,j=0}^{n-2}$ can be computed by

$$\alpha_{ij} = \alpha_{ji} = (-1)^{i+j} \frac{\beta_{n-1-i}\,\beta_j}{\beta_{n-1}}, \qquad i \geq j$$

with

$$\beta_{i-1} = \frac{1}{2\sqrt{8}} \left((3+\sqrt{8})^i - (3-\sqrt{8})^i \right)$$

which can be seen by explicitly writing down the forward and backward elimination. In each row of $\mathcal{A}^{-1}$, the diagonal element dominates the other coefficients and from the standard bound of the spectral radius

$$\max_{\|\mathbf{x}\|=1} \mathbf{x}^T \mathcal{A}^{-1} \mathbf{x} \leq \frac{1}{4}$$

we see

$$\left| (0,\dots,1,\dots,0)\, \mathcal{A}^{-1} (0,\dots,1,\dots,0)^T \right| = |\alpha_{ii}| \leq \frac{1}{4}.$$

Thus for each coefficient $|\alpha_{ij}| \leq \frac{1}{4}$.

For $m \geq 1$ the components with odd indices on the right side of (12) vanish and the dimension $n-1$ of $\mathcal{A}$ is odd. We have

$$\mathcal{A}\,(-1,3,-1,3,\dots,-1,3,-1)^T = (-3,16,0,16,\dots,0,16,-3)^T$$

or

$$\mathcal{A}^{-1}\,(-3,16,0,16,\dots,0,16,-3)^T = (-1,3,-1,3,\dots,-1,3,-1)^T$$

and therefore

$$\sum_{j=0}^{\frac{n}{2}-2} \alpha_{i,2j+1} = \begin{cases} \frac{1}{16}\Big(3\,(\alpha_{i,0}+\alpha_{i,n-2})-1\Big) & \text{for} \quad i \text{ even} \\ \frac{1}{16}\Big(3\,(\alpha_{i,0}+\alpha_{i,n-2})+3\Big) & \phantom{\text{for}} \quad i \text{ odd.} \end{cases}$$

Notice that all coefficients $\alpha_{i,2j+1}$ have the same sign. From $|\alpha_{ij}| \leq \frac{1}{4}$ we finally obtain

$$\sum_{j=0}^{\frac{n}{2}-2} |\alpha_{i,2j+1}| \;\leq\; \frac{3}{16}.$$

The $\triangle^4 \mathbf{p}_{2i}^{m+1}$ are computed by

$$\triangle^4 \mathbf{p}_{2i}^{m+1} \;=\; \sum_{j=0}^{\frac{n}{2}-2} \alpha_{i,2j+1} \triangle^4 \mathbf{p}_{2j}^{m}.$$

Hence, it follows

$$\|\triangle^4\,\mathbf{P}_{m+1}\|_\infty \;\leq\; \frac{3}{16}\;\|\triangle^4\,\mathbf{P}_m\|_\infty$$

or

$$\|2^{2m}\,\triangle^4\,\mathbf{P}_m\|_\infty \;=\; O\Big(\Big(\frac{3}{4}\Big)^m\Big). \qquad \square$$

7 Local refinement schemes

By now we only considered refinement schemes which are based on a *global* optimization problem. In order to construct local refinement schemes we can restrict the optimization to some local subpolygon. This means a new point $\mathbf{p}_{2l+1}^{m+1}$ is computed by minimizing some energy functional over a *window* $\mathbf{p}_{l-r}^{m}, \ldots, \mathbf{p}_{l+1+r}^{m}$. As the index l varies, the window is shifted in the same way.

Let E be a given quadratic energy functional. The solution of its minimization over the window $\mathbf{p}_{l-r}^{m}, \ldots, \mathbf{p}_{l+1+r}^{m}$ is computed by solving an Euler-Lagrange-equation

$$\mathcal{B}\,(\mathbf{p}_{2l+1+2i}^{m+1})_{i=-r}^{r} = \mathcal{C}\,(\mathbf{p}_{l+i}^{m})_{i=-r}^{r+1}. \tag{13}$$

The matrix of $\mathcal{B}^{-1}\,\mathcal{C}$ can be computed explicitly and the weight coefficients by which a new point $\mathbf{p}_{2l+1}^{m+1}$ is computed, can be read off from the corresponding row in $\mathcal{B}^{-1}\,\mathcal{C}$. Since the coefficients depend on E and r only, this construction yields a stationary refinement scheme.

For such local schemes the convergence analysis is independent from the topological structure (open/closed) of the polygons to be refined. The formalisms of (Cavaretta et al., 1991), (Dyn & Levin, 1990) or (Kobbelt, 1995b) can be applied.

Minimizing the special energy functional $E_k(\mathbf{P})$ from (7) over open polygons allows the interesting observation that the resulting refinement scheme has polynomial precision of degree $k-1$. This is obvious since for points lying equidistantly parameterized on a polynomial curve of degree $k-1$, all k-th differences vanish and $E_k(\mathbf{P}) = 0$ clearly is the minimum of the quadratic functional.

Since the $2r+2$ points which form the subpolygon $\mathbf{p}^m_{l-r}, \ldots, \mathbf{p}^m_{l+1+r}$ uniquely define an interpolating polynomial of degree $2r+1$, it follows that the local schemes based on the minimization of $E_k(\mathbf{P})$ are identical for $k \geq 2r+2$. These schemes coincide with the Lagrange-schemes of (Deslauriers & Dubuc, 1989). Notice that $k \leq 4r+2$ is necessary because higher differences are not possible on the polygon $\mathbf{p}^{m+1}_{2(l-r)}, \ldots, \mathbf{p}^{m+1}_{2(l+1+r)}$ and minimizing $E_k(\mathbf{P}) \equiv 0$ makes no sense.

The local variational schemes provide a nice feature for practical purposes. One can use the refinement rules defined by the coefficients in the rows of $\mathcal{B}^{-1}\,\mathcal{C}$ in (13) to compute points which subdivide edges near the ends of open polygons. Pure stationary refinement schemes do not have this option and one therefore has to deal with *shrinking ends*. This means one only subdivides those edges which allow the application of the given subdivision mask and cuts off the remaining part of the unrefined polygon.

If $k \geq 2r+2$ then the use of these auxiliary rules causes the limiting curve to have a polynomial segment at both ends. This can be seen as follows. Let $\mathbf{P}_0 = (\mathbf{p}^0_0, \ldots, \mathbf{p}^0_n)$ be a given polygon and denote the polynomial of degree $2r+1 \leq k-1$ uniformly interpolating the points $\mathbf{p}^0_0, \ldots, \mathbf{p}^0_{2r+1}$ by $f(x)$.

The first vertex of the refined polygon $\mathbf{P}_1$ which not necessarily lies on $f(x)$ is $\mathbf{p}^1_{2r+3}$. Applying the same refinement scheme iteratively, we see that if $\mathbf{p}^m_{\delta_m}$ is the first vertex of $\mathbf{P}_m$ which does not lie on $f(x)$ then $\mathbf{p}^{m+1}_{\delta_{m+1}} = \mathbf{p}^{m+1}_{2\delta_m - 2r-1}$ is the first vertex of $\mathbf{P}_{m+1}$ with this property. Let $\delta_0 = 2r+2$ and consider the sequence

$$\lim_{m\to\infty} \frac{\delta_m}{2^m} \;=\; (2r+2) - (2r+1) \lim_{m\to\infty} \sum_{i=1}^{m} 2^{-i} \;=\; 1.$$

Hence, the limiting curve $\mathbf{P}_\infty$ has a polynomial segment $f(x)$ between the points $\mathbf{p}^0_0$ and $\mathbf{p}^0_1$. An analog statement holds at the opposite end between $\mathbf{p}^0_{n-1}$ and $\mathbf{p}^0_n$.

This feature also arises naturally in the context of Lagrange-schemes where the new points near the ends of an open polygon can be chosen to lie on the first or last well-defined polynomial. It can be used to exactly compute the derivatives at the endpoints $\mathbf{p}^0_0$ and $\mathbf{p}^0_n$ of the limiting curve and it also provides the possibility to smoothly connect refinement curves and polynomial splines.

8 Computational Aspects

Since for the variational refinement schemes the computation of the new points $\mathbf{p}_{2i+1}^{m+1}$ involves the solution of a linear system, the algorithmic structure of these schemes is slightly more complicated than it is in the case of stationary refinement schemes. However, for the refinement of an open polygon $\mathbf{P}_m$ the computational complexity is still linear in the length of $\mathbf{P}_m$. The matrix of the system that has to be solved, is a banded Toeplitz-matrix with a small number of pertubations at the boundaries.

In the closed polygon case, the best we can do is to solve the circulant system in the Fourier domain. In particular, we transform the initial polygon $\mathbf{P}_0$ once and then perform m refinement steps in the Fourier domain where the convolution operator becomes a diagonal operator. The refined spectrum $\widehat{\mathbf{P}}_m$ is finally transformed back in order to obtain the result $\mathbf{P}_m$. The details can be found in (Kobbelt, 1995c). For this algorithm, the computational costs are dominated by the discrete Fourier transformation of $\widehat{\mathbf{P}}_m$ which can be done in $O(n \log(n)) = O(2^m\, m)$ steps. This is obvious since the number $n = 2^m\, n_0$ of points in the refined polygon $\mathbf{P}_m$ allows to apply m steps of the fast Fourier transform algorithm.

The costs for computing $\mathbf{P}_m$ are therefore $O(m)$ per point compared to $O(1)$ for stationary schemes. However, since in practice only a small number of refinement steps are computed, the constant factors which are hidden within these asymptotic estimates are relevant. Thus, the fact that implicit schemes need a smaller bandwidth than stationary schemes to obtain the same differentiability of the limiting curve (cf. Table 1) equalizes the performance of both.

In the implementation of these algorithms it turned out that all these computational costs are dominated by the 'administrative' overhead which is necessary, e.g., to build up the data structures. Hence, the differences in efficiency between stationary and implicit refinement schemes can be neglected.

References

Cavaretta, A. and Dahmen, W. and Micchelli, C. (1991), Stationary Subdivision, Memoirs of the AMS 93, 1–186

Clegg, J. (1970), *Variationsrechnung*, Teubner Verlag, Stuttgart

Deslauriers, G. and Dubuc, S. (1989), Symmetric iterative interpolation processes, Constructive Approximation 5, 49–68

Dubuc, S. (1986), Interpolation through an iterative scheme, Jour. of Mathem. Anal. and Appl. 114, 185–204

Dyn, N. and Gregory, J. and Levin, D. (1987), A 4-point interpolatory subdivision scheme for curve design, CAGD 4, 257–268

Dyn, N. and Levin, D. (1990), Interpolating subdivision schemes for the generation of curves and surfaces, in: Haußmann W. and Jetter K. eds., *Multivariate Approximation* *and Interpolation*, Birkhäuser Verlag, Basel

Dyn, N. and Levin, D. and Liu, D. (1992), Interpolatory convexity-preserving subdivision schemes for curves and surfaces, CAD 24, 221–216

Dyn, N. (1991), Subdivision schemes in computer aided geometric design, in: Light, W. ed., *Advances in Numerical Analysis II, Wavelets, Subdivisions and Radial Functions*, Oxford University Press

Golub, G. and Van Loan, C. (1989), *Matrix Computations*, John Hopkins University Press

Kobbelt, L. (1995a), *Iterative Erzeugung glatter Interpolanten*, Universität Karlsruhe

Kobbelt, L. (1995b), Using the discrete Fourier-transform to analyse interpolatory subdivision schemes, Preprint

Kobbelt, L. (1995c), Interpolatory Refinement is Low Pass Filtering, in Daehlen, M. and Lyche, T. and Schumaker, L. eds., Math. Meth in CAGD III

Meier, H. and Nowacki, H. (1987), Interpolating curves with gradual changes in curvature, CAGD 4, 297–305

Le Méhauté A. and Utreras, F. (1994), Convexity-preserving interpolatory subdivision, CAGD 11, 17–37

Paluszny M. and Prautzsch H. and Schäfer, M. (1994), Corner cutting and interpolatory refinement, Preprint

Sapidis, N. (1994), *Designing Fair Curves and Surfaces*, SIAM, Philadelphia

Widom, H. (1965), Toeplitz matrices, in: Hirschmann, I. ed., *Studies in Real and Complex Analysis*, MAA Studies in Mathematics 3