Using the Discrete Fourier-Transform to Analyze the Convergence of Subdivision Schemes

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Abstract

While the *continuous* Fourier transform is a well-established standard tool for the analysis of subdivision schemes, we present a new technique based on the *discrete* Fourier transform instead. We first prove a very general convergence criterion for arbitrary interpolatory schemes, i.e., for non-stationary, globally supported or even non-linear schemes. Then we use the discrete Fourier transform as an algebraic tool to transform subdivision schemes into a form suitable for the analysis. This allows us to formulate simple and numerically stable sufficient criteria for the convergence of subdivision schemes of very general type. We analyze some example schemes to illustrate the resulting easy-to-apply criteria which merely require to numerically estimate the maximum of a smooth function on a compact interval.

1 Introduction

Univariate subdivision schemes are usually defined as the iterative application of an operator which maps a given polygon $\mathbf{P}_m = [\mathbf{p}_i^m]$ to a refined polygon $\mathbf{P}_{m+1} = [\mathbf{p}_i^{m+1}]$. Such an operator is given by two rules for computing the new vertices $\mathbf{p}_{2i}^{m+1}$ and $\mathbf{p}_{2i+1}^{m+1}$ depending on the parity of their index. In the important special case of interpolatory refinement (cf. Fig. 1), the first rule reduces to the interpolation condition $\mathbf{p}_{2i}^{m+1} = \mathbf{p}_i^m$.

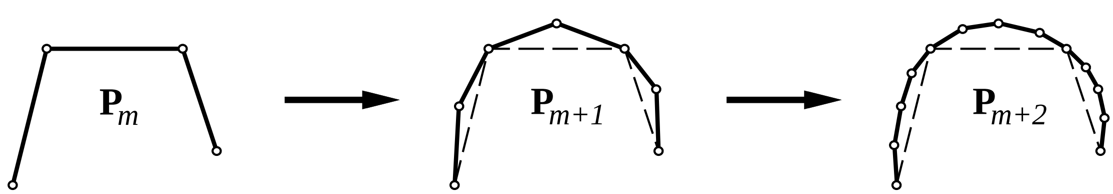

Figure 1: Interpolatory refinement

If the new vertices are chosen appropriately in every refinement step, then the sequence of polygons $[\mathbf{P}_m]$ may converge to a smooth limit curve $\mathbf{P}_\infty$. One is mainly interested in techniques for *constructing* and *analyzing* specific refinement schemes which produce limit curves of a certain smoothness. Although (analytic) differentiability does not necessarily imply (geometric) smoothness, one usually measures the quality of $\mathbf{P}_\infty$ by the quality of its coordinate functions, i.e., by the number of continuous derivatives with respect to a uniform parameterization. The additional requirement that the parameterization must not become singular in the limit is neglected to simplify the analysis.

There are many approaches to the definition of rules which iteratively smooth the shape of a given polygon. In [CDM91], [Dyn91] and [Rio92] the class of *stationary subdivision schemes* is extensively investigated. These schemes are defined by a *finite* sequence of coefficients $[\alpha_k]$. The new vertices $\mathbf{p}_i^{m+1}$ are computed by fixed linear combinations:

$$\mathbf{p}_i^{m+1} \;=\; \sum_{k\,\in\,\mathbb{Z}} \alpha_{i-2k}\; \mathbf{p}_k^m. \tag{1}$$

The most famous representatives of this class are the subdivision algorithm for B-Splines [LR80] and the interpolatory Lagrange schemes [DD89].

A more general approach allows the linear combinations to vary in every refinement step, i.e., to use coefficients α_k^m. These schemes are called *non-stationary* [DL94]. Further, one can vary the weights with the index i, i.e., $\alpha_{i,k}^m$ which leads to *non-uniform schemes* [Qu92], [War95].

A fairly general class of (potentially) *globally* supported schemes are the *implicit refinement schemes*. These are interpolatory schemes where the new vertices of the refined polygon $\mathbf{P}_{m+1}$ are implicitly given by

$$\forall\, i: \qquad \mathbf{p}_{2i}^{m+1} \;=\; \mathbf{p}_i^m \qquad \text{and} \qquad \sum_{k\,\in\,\mathbb{Z}} \beta_k\; \mathbf{p}_{2i+1+k}^{m+1} \;=\; 0 \tag{2}$$

with an arbitrary (but usually symmetric) finite sequence of coefficients $[\beta_k]$. Such schemes naturally arise in the context of variational refinement schemes which determine the new vertices $\mathbf{p}_{2i+1}^{m+1}$ in the refined polygon $\mathbf{P}_{m+1}$ by minimizing some quadratic energy functional while holding the even indexed vertices fixed [Kob96]. Notice that for $\beta_{2k} = \delta_{k,0}$, the scheme (2) is stationary.

In the context of signal processing, subdivision schemes arise as one component of a multi-resolution analysis [Dau92], [HV93]. The discrete multi-resolution set-up provides a set of filters for analyzing and reconstructing a given real-valued sequence. Here, *analyzing* means to split the given sequence into several frequency bands and *reconstruction* means the recombination of those bands. A subdivision scheme corresponds to the application of the reconstruction filter to a decomposition where all but the lowest band vanish identically (i.e., have zero energy).

To simplify the notation for processing finite sequences $[\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}]$, one usually extends them to a bi-infinite sequence $[\mathbf{p}_i]$ with $i \in \mathbb{Z}$ by adding infinitely many zero-elements on both sides. This construction topologically corresponds to open polygons and naturally leads to the application of the continuous Fourier-transform as a major tool for the analysis. Alternatively, we can as well assume $[\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}]$ to be a closed polygon and extend the input sequence by periodicity, i.e., by $\mathbf{p}_{i+n} = \mathbf{p}_i$. In the periodic setting, the discrete Fourier-transform turns out to be more appropriate for the analysis.

However, in most of the literature the continuous Fourier transform is applied which allows the convergence analysis of subdivision schemes like (1) along the following lines (cf. [Dau92]). The vertices of the polygons $\mathbf{P}_m$ are indexed by $\mathbb{Z}$ and, by associating the vertex $\mathbf{p}_i^m$ with the parameter value $t_i^m = i\,2^{-m}$, one defines a sequence of functions

$$\mathbf{P}_m(x) \;=\; \sum_{i\,\in\,\mathbb{Z}} \mathbf{p}_i^m \;\Phi(2^m\,x - i) \tag{3}$$

on the real line, with the basis function Φ usually not known explicitly. If the subdivision scheme (1) is convergent then the smoothness of its limit curves can be determined by analyzing the basis function Φ as the unique solution of the refinement equation

$$\Phi(x) \;=\; \sum_{i\,\in\,\mathbb{Z}} \alpha_i \;\Phi(2x - i).$$

The Fourier transform yields

$$\hat{\Phi}(y) \;=\; \alpha(y/2)\;\hat{\Phi}(y/2)$$

with

$$\alpha(y) \;=\; \frac{1}{2}\sum_{i\,\in\,\mathbb{Z}} \alpha_i\; e^{-j\,y\,i} \tag{4}$$

and the recursive expansion

$$\hat{\Phi}(y) \;=\; \hat{\Phi}(0)\;\prod_{k=1}^{\infty} \alpha(y\,2^{-k}).$$

The rate of decay of $\hat{\Phi}(y)$ for $|y| \to \infty$ is a sufficient condition for the (Hölder-) differentiability of Φ and therefore for the smoothness of the curves generated by the subdivision scheme. There are several ways to estimate this decay from certain properties of $\alpha(y)$ [Dau92]. Notice that the scaling factor of 2^{-k} in the infinite product means that the trigonometric polynomial $\alpha(y\,2^{-k})$ is more and more stretched out. In fact, $\hat{\Phi}$ is no longer periodic.

In contrast to this classic technique, we propose to do the convergence analysis not by considering the functions in the Fourier-domain over the whole real line but by restricting the spectrum to a finite interval and use the discrete Fourier transform. However, we will not use it as a specialization of the above technique to periodic and discrete distributions $\mathbf{P}_m$ but we consider this transformation merely as a basis transformation of the *vector* $\mathbf{P}_m$ into a basis which consists of the special eigenvectors common to all circulant matrices. Thus the only fact we are exploiting is the *algebraic* property that the discrete Fourier transform maps convolution operators to diagonal operators.

Staying within the functional set-up, our application of the discrete Fourier transform would correspond to the use of periodic functions

$$\mathbf{P}_m(x) \;=\; \sum_{i=0}^{n2^m-1} \mathbf{p}_i^m \;\Delta_m(x-i)$$

with

$$\Delta_m(x) \;:=\; \sum_{k\,\in\,\mathbb{Z}} \delta(x - k\,n\,2^m),$$

i.e., the vertex $\mathbf{p}_i^m$ is associated with the parameter value $t_i^m = i \bmod n\,2^m$. This *stretching* in the time domain (compared to (3)) causes a shrinking of the corresponding discrete spectrum which therefore remains periodic:

$$\widehat{\mathbf{P}}_m(y) \;=\; \Big(\sum_{i=0}^{n2^m-1} \mathbf{p}_i^m \, e^{-2\,\pi\,j\,y\,i}\Big)\;\Big(\sum_{k\,\in\,\mathbb{Z}} \delta(y - k/n\,2^m)\Big).$$

Hence, the analysis in the frequency domain can be restricted to sampling equidistant points on the unit interval instead of estimating the asymptotic decay.

This paper is organized as follows: In Section 3, we give an elementary proof (without using any Fourier-technique) for a very general convergence theorem (Theorem 5) which applies to arbitrary interpolatory schemes. Although this theorem makes a statement about the differentiability of the limit curves $\mathbf{P}_\infty$ with respect to a specific parameterization, the sufficient condition can be considered without explicitly referring to this parameterization, i.e., the theorem can be understood as a plain numerical criterion which guarantees smoothness in some (not further specified) sense. Hence, for the analysis we do not have to interpret the vertices $\mathbf{p}_i^m$ to be function values of the *function* $\mathbf{P}_m$ but we can consider the $\mathbf{p}_i^m$ as components of the *vector* $\mathbf{P}_m$. In this context, the pseudo-parameterization $t_i^m = i \bmod n\,2^m$ does make sense and motivates the use of the discrete Fourier-transform for the manipulation of the vector $\mathbf{P}_m$ in the later sections.

We will apply Theorem 5 mainly to implicit refinement schemes (cf. Section 4) but since the Theorem 5 is known to hold for non-interpolatory *stationary* schemes as well [Dyn91], these schemes can be analyzed, too (cf. Section 6).

In Section 5 the main results of this paper (Theorem 7) are derived by combining the general convergence Theorem 5 with the discrete Fourier transform as an algebraic tool to rewrite the subdivision operation in a convenient form. In Section 7 several interesting examples for the application of this approach are given.

2 The discrete Fourier-transform

The fundamental tool which is used in this paper is the DFT. Since there are multiple ways of defining this transformation, we briefly collect the most important facts needed here.

Let $\omega_n = e^{-2\pi j/n}$ be an nth root of unity. Then the DFT of a n-dimensional vector $\mathbf{P} = [\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}]$ is defined to be $\widehat{\mathbf{P}} := [\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{n-1}]$ with

$$\hat{\mathbf{p}}_i \; := \; \sum_{k=0}^{n-1} \omega_n^{ik} \, \mathbf{p}_k.$$

The vector $\widehat{\mathbf{P}}$ can be considered as the discrete frequency spectrum of $\mathbf{P}$. Therefore we call the space where $\mathbf{P}$ lives the *space* or *time domain* and $\widehat{\mathbf{P}}$ lives in the *frequency domain*. The inverse transformation is

$$\mathbf{p}_i \; = \; \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{-ik} \, \hat{\mathbf{p}}_k,$$

as can be verified easily. The latter equation yields a simple estimate for the maximum component of $\mathbf{P}$

$$\|\mathbf{P}\|_\infty \; \leq \; \frac{1}{n} \, \|\widehat{\mathbf{P}}\|_1. \tag{5}$$

One of the central results concerning the DFT is called the *convolution theorem*. Given a finite convolution operator $\mathcal{A} = \mathrm{circ}[\alpha_0, \ldots, \alpha_{n-1}]^T$ and a vector $\mathbf{P} = [\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}]$ the Fourier transform of $\mathbf{Q} = \mathcal{A}\,\mathbf{P}$ has the components

$$\hat{\mathbf{q}}_l \; = \; \sum_{k=0}^{n-1} \omega_n^{kl} \sum_{i=0}^{n-1} \alpha_{k-i} \, \mathbf{p}_i \; = \; \sum_{i=0}^{n-1} \omega_n^{il} \, \mathbf{p}_i \sum_{k=0}^{n-1} \omega_n^{(k-i)\,l} \, \alpha_{k-i} \; = \; \hat{\alpha}_l \, \hat{\mathbf{p}}_l,$$

where the index of α_k is taken *modulo* n. Thus, the convolution operator $\mathcal{A}$ in the space domain corresponds to the diagonal operator $\widehat{\mathcal{A}} = \mathrm{diag}[\hat{\alpha}_0, \ldots, \hat{\alpha}_{n-1}]$ in the frequency domain[1].

The difference operator $\triangle$ is used for discrete approximations of differentiation. With

$$\triangle^k \mathbf{p}_i \; := \; \sum_{l=0}^{k} \binom{k}{l} \, (-1)^{k+l} \, \mathbf{p}_{i+l},$$

we define the *cyclic difference polygon* $\triangle^k \mathbf{P} := [\triangle^k \mathbf{p}_i]_{i=0}^{n-1}$ having forward difference vectors as its components. The corresponding *difference spectrum* in the frequency domain is

$$\triangle^k \widehat{\mathbf{P}} \; := \; \Big[(\omega_n^{-i} - 1)^k \, \hat{\mathbf{p}}_i \Big]_{i=0}^{n-1}. \tag{6}$$

A more detailed description of the DFT can be found in, e.g., [BBN87], [OS89].

[1] Notice that the transpose operation in the definition of $\mathcal{A}$ causes a "reversing" of the coefficients $\alpha_1, \ldots, \alpha_{n-1}$ in each row of the circulant matrix.

3 Convergence criteria for sequences of polygons

Let $[\mathbf{P}_m]$ be a sequence of open polygons generated by the iterative application of an arbitrary interpolatory refinement scheme to the given polygon $\mathbf{P}_0 = [\mathbf{p}_0^0, \ldots, \mathbf{p}_n^0]$. We prove sufficient conditions for this sequence to have a well defined limit curve $\mathbf{P}_\infty$ which is k-times continuously differentiable with respect to a uniform parameterization. These criteria will not use any specific properties of the refinement scheme other than the interpolation property $\mathbf{p}_{2i}^{m+1} = \mathbf{p}_i^m$. Consequently they apply to global, non-stationary and even to non-linear schemes.

Since we are only dealing with sequences of polygons and are not analyzing specific refinement operators, we can consider closed polygons as a special case of finite open polygons with periodic behavior. This can be done due to the fact that differentiability is a *local* property. By a similar argument the results of this section can be generalized to work for infinite polygons as well. Hence, throughout *this* section, we consider refinement operators which map $\mathbb{R}^{n+1\times d}$ to $\mathbb{R}^{2n+1\times d}$ with d the dimension of the space where the polygon's vertices lie.

To make the concept of differentiability meaningful for the limit of a sequence of polygons, we have to choose a parameterization for this curve. In order to assign the same parameter value t_i^m to identical vertices $\mathbf{p}_{i\,2^r}^{m+r}$ in different polygons $\mathbf{P}_{m+r}$, we choose the equidistant nodes

$$t_i^m \;=\; i\,2^{-m}.$$

Thus, each polygon $\mathbf{P}_m$ should be considered as a piecewise linear function $\mathbf{P}_m(x)$ with $\mathbf{P}_m(i\,2^{-m}) = \mathbf{p}_i^m$.

If $\mathbf{P}_m = [\mathbf{p}_0^m, \ldots, \mathbf{p}_{n\,2^m}^m]$ is finite and open, then $\triangle^k \mathbf{P}_m$ consists of only $(n\,2^m + 1 - k)$ components. Since some arguments we apply during the proofs require all difference polygons to be defined over the same compact parameter interval, we chose a uniform parameterization with step-width

$$h_{m,k} \;=\; \frac{n}{n\,2^m - k} \tag{7}$$

for the difference polygons. The polygon $\triangle^k \mathbf{P}_m$ then corresponds to a piecewise linear function for which $\triangle^k \mathbf{P}_m(i\,h_{m,k}) = \triangle^k \mathbf{p}_i^m$ and the $\triangle^k \mathbf{P}_m$ live over the same parameter interval $[0, n]$ for all k and m.

The first lemma yields a useful characterization of difference polygons for sequences $[\mathbf{P}_m]$ which satisfy the interpolation condition $\mathbf{p}_{2i}^{m+1} = \mathbf{p}_i^m$. Since we consider the iterative refinement of finite open polygons starting with $\mathbf{P}_0 = [\mathbf{p}_0^0, \ldots, \mathbf{p}_n^0]$, we obtain $\mathbf{P}_m = [\mathbf{p}_0^m, \ldots, \mathbf{p}_{n\,2^m}^m]$ after m refinement steps.

Lemma 1 *Let $[\mathbf{P}_m]$ be a sequence of polygons. The scheme by which they are generated is an interpolatory refinement scheme if and only if, for all $m, k \in \mathbb{N}$, the condition*

$$\triangle^k p_i^m = \sum_{l=0}^{k} \binom{k}{l} \triangle^k p_{2i+l}^{m+1}, \qquad i = 0, \ldots, n\,2^m - k$$

holds.

Proof The sufficient part of the proof is trivial since setting $k = 0$ reproduces the definition of interpolatory refinement schemes. The necessary part is also obvious for $k = 0$. The general statement is proved by induction. If the statement holds for some value k, then

$$\begin{aligned}
\triangle^{k+1} p_i^m &= \triangle^k p_{i+1}^m - \triangle^k p_i^m \\
&= \sum_{l=0}^{k} \binom{k}{l} (\triangle^k p_{2i+2+l}^{m+1} - \triangle^k p_{2i+1+l}^{m+1} + \triangle^k p_{2i+1+l}^{m+1} - \triangle^k p_{2i+l}^{m+1}) \\
&= \sum_{l=0}^{k} \binom{k}{l} (\triangle^{k+1} p_{2i+1+l}^{m+1} + \triangle^{k+1} p_{2i+l}^{m+1}) \\
&= \sum_{l=0}^{k+1} \binom{k+1}{l} \triangle^{k+1} p_{2i+l}^{m+1}. \qquad \square
\end{aligned}$$

The next lemma relates successive difference polygons in the sequence $[\triangle^k \mathbf{P}_m]$.

Lemma 2 *Let $[\mathbf{P}_m]$ be a sequence of polygons generated by the iterative application of an interpolatory refinement scheme. Then there exists a constant σ which only depends on k such that*

$$\|\triangle^k \mathbf{P}_m - 2^k \, \triangle^k \mathbf{P}_{m+1}\|_\infty \;\leq\; \sigma \|\triangle^{k+1} \mathbf{P}_{m+1}\|_\infty.$$

Proof The maximum distance is obviously obtained at some vertex. The polygons $\triangle^k \mathbf{P}_m$ and $\triangle^k \mathbf{P}_{m+1}$ are parameterized with different step-widths but the distance between a vertex V of one polygon and a point on some edge E of the other polygon can be bounded by the maximum of the distances between the vertex V and the two endpoints of E.

For the parameter values of $\triangle^k \mathbf{p}_{2i}^{m+1}$, $\triangle^k \mathbf{p}_i^m$ and $\triangle^k \mathbf{p}_{2i+k}^{m+1}$, we have

$$\frac{2\,i\,n}{n\,2^{m+1} - k} \leq \frac{i\,n}{n\,2^m - k} \leq \frac{(2\,i + k)\,n}{n\,2^{m+1} - k}, \qquad i = 0, \ldots, n\,2^m - k.$$

Thus, it is sufficient to consider the distances between $\triangle^k \mathbf{p}_i^m$ and $2^k \, \triangle^k \mathbf{p}_{2i+r}^{m+1}$ for $r = 0, \ldots, k$. Lemma 1 implies

$$\begin{aligned}
|\triangle^k \mathbf{p}_i^m - 2^k \, \triangle^k \mathbf{p}_{2i+r}^{m+1}| &= \left| \sum_{l=0}^{k} \binom{k}{l} \triangle^k \mathbf{p}_{2i+l}^{m+1} - 2^k \, \triangle^k \mathbf{p}_{2i+r}^{m+1} \right| \\
&\leq \sigma \, \|\triangle^{k+1} \mathbf{P}_{m+1}\|_\infty
\end{aligned}$$

since the binomial coefficients sum to 2^k. $\square$

The next lemma reveals a correlation between the asymptotic behavior of different difference polygons.

Lemma 3 *Let $[\mathbf{P}_m]$ be a sequence of polygons generated by the iterative application of an interpolatory refinement scheme. If there exists a $q < 2^k$ such that*

$$\sum_{m=0}^{\infty} \|q^m \triangle^{k+1} \mathbf{P}_m\|_\infty < \infty,$$

then

$$\sum_{m=0}^{\infty} \|q^m \triangle^{k} \mathbf{P}_m\|_\infty < \infty.$$

Proof From Lemma 2 it follows that there exists some σ such that

$$\begin{aligned}
\|\triangle^k \mathbf{P}_m\|_\infty &\le 2^{-k} \|\triangle^k \mathbf{P}_{m-1}\|_\infty + \sigma \|\triangle^{k+1} \mathbf{P}_m\|_\infty \\
&\le 2^{-2k} \|\triangle^k \mathbf{P}_{m-2}\|_\infty + 2^{-k} \sigma \|\triangle^{k+1} \mathbf{P}_{m-1}\|_\infty + \sigma \|\triangle^{k+1} \mathbf{P}_m\|_\infty \\
&\vdots \\
&\le 2^{-mk} \|\triangle^k \mathbf{P}_0\|_\infty + \sigma \sum_{i=1}^{m} 2^{(i-m)k} \|\triangle^{k+1} \mathbf{P}_i\|_\infty .
\end{aligned}$$

Setting $r := q\, 2^{-k} < 1$ we obtain for every $N \in \mathbb{N}$

$$\begin{aligned}
\sum_{m=0}^{N} \|q^m \triangle^k \mathbf{P}_m\|_\infty &\le \sum_{m=0}^{N} r^m \|\triangle^k \mathbf{P}_0\|_\infty + \sigma \sum_{m=1}^{N} \sum_{i=1}^{m} r^{m-i} \|q^i \triangle^{k+1} \mathbf{P}_i\|_\infty \\
&\le \frac{1}{1-r} \|\triangle^k \mathbf{P}_0\|_\infty + \sigma \sum_{m=1}^{N} \Big(\sum_{i=0}^{N-1} r^i \Big) \|q^m \triangle^{k+1} \mathbf{P}_m\|_\infty \\
&\le \frac{1}{1-r} \Big(\|\triangle^k \mathbf{P}_0\|_\infty + \sigma \sum_{m=1}^{N} \|q^m \triangle^{k+1} \mathbf{P}_m\|_\infty \Big),
\end{aligned}$$

Now taking $N \to \infty$ yields

$$\sum_{m=0}^{\infty} \|q^m \triangle^k \mathbf{P}_m\|_\infty \le \frac{1}{1-r} \Big(\|\triangle^k \mathbf{P}_0\|_\infty + \sigma \sum_{m=1}^{\infty} \|q^m \triangle^{k+1} \mathbf{P}_m\|_\infty \Big) < \infty. \qquad \square$$

The next lemma justifies the use of the difference operator $\triangle$ as a discrete approximation to the differential operator.

Lemma 4 *Let $[\mathbf{P}_m]$ be a sequence of polygons generated by the iterative application of an interpolatory refinement scheme. Then*

$$\lim_{m\to\infty} \mathbf{P}_m = g \in C^k, \quad \frac{d^k}{dx^k}\, g = f \qquad \Longleftrightarrow \qquad \lim_{m\to\infty} 2^{km} \triangle^k \mathbf{P}_m = f \in C^0.$$

Proof For the implication from left to right, we use the fact that all intermediate vertices $\mathbf{p}_i^m$ already lie on the limit curve $\mathbf{P}_\infty$, i.e., $\mathbf{P}_\infty(i\,2^{-m}) = g(i\,2^{-m}) = \mathbf{p}_i^m$. Due to the special parameterization of the difference polygons $\triangle^k \mathbf{P}_m$ with step-width (7) we have $\triangle^k \mathbf{P}_m(i\,h_{m,k}) = \triangle^k g(i\,2^{-m})$ with $i\,2^{-m} \le i\,h_{m,k} \le (i+k)\,2^{-m}$ and the statement follows from the application of the difference operator to the Taylor expansion of g and letting $k\,2^{-m}$ tend to zero.

More precisely, since the $\triangle^k$-operator kills all polynomials up to the degree of $k-1$, the existence of a Taylor-expansion for g at $x = i\,2^{-m}$ implies that for some $\xi_l \in [i\,2^{-m}, (i+l)\,2^{-m}]$

$$2^{km}\,\triangle^k \mathbf{P}_m(i\,h_{m,k}) \;=\; \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{k+l}\,l^k}{k!}\, g^{(k)}(\xi_l)$$

where the weight coefficients sum to unity and thus

$$|2^{km}\,\triangle^k \mathbf{P}_m(i\,h_{m,k}) - f(i\,h_{m,k})| \;\le\; \sum_{l=0}^{k} \binom{k}{l} \frac{l^k}{k!}\, |f(\xi_l) - f(i\,h_{m,k})|.$$

Due to the uniform continuity of f which is a continuous function on the compact interval $[0,n]$, the right-hand side of this inequation can be made smaller than any ε by choosing m independently of i. Further, if m is such that even $|f(x) - f(y)| < \varepsilon$ for $|x - y| \le h_{m,k}$ then

$$\max_{x\,\in\,[0,n]} |\,2^{km}\,\triangle^k\,\mathbf{P}_m(x) - f(x)| \le 2\,\varepsilon.$$

The opposite implication is shown by induction over k. The case $k = 0$ is trivial. Let $[2^{(k+1)m}\triangle^{k+1}\mathbf{P}_m]$ be a converging sequence with continuous limit f which is bounded on the compact interval $[0,n]$. Then the sequence $\|2^{km}\triangle^{k+1}\mathbf{P}_m\|_\infty$ converges like $O(2^{-m})$ and, by Lemma 2, the polygons $2^{km}\triangle^k\mathbf{P}_m$ form a Cauchy sequence of continuous functions with continuous limit $\tilde{g}$.

Let $\mathbf{Q}_m : [0,n] \to \mathbb{R}$ be piecewise constant functions over the intervals $(i\,h_{m,k}, (i+1)\,h_{m,k}]$ with function values

$$\mathbf{Q}_m(i\,h_{m,k}) \;:=\; \frac{2^{km}}{h_{m,k}} \triangle^{k+1}\mathbf{p}_{i-1}^m, \qquad i = 1, \ldots, n\,2^m - k.$$

The maximum distance between $\mathbf{Q}_m$ and $2^{(k+1)m}\triangle^{k+1}\mathbf{P}_m$ is

$$
\begin{aligned}
\|\mathbf{Q}_m - 2^{(k+1)\,m}\triangle^{k+1}\mathbf{P}_m\|_\infty &\leq \left|\frac{1}{h_{m,k}} - 2^m\right| \|\, 2^{km}\,\triangle^{k+1}\mathbf{P}_m\,\|_\infty + \|\, 2^{(k+1)\,m}\triangle^{k+2}\mathbf{P}_m\,\|_\infty \\
&= \frac{k}{n}\,O(2^{-m}) + \|\, 2^{(k+1)\,m}\triangle^{k+2}\mathbf{P}_m\,\|_\infty,
\end{aligned}
$$

and therefore the sequence $\mathbf{Q}_m$ also uniformly converges to the limit f since $2^{(k+1)m}\triangle^{k+2}\mathbf{P}_m$ uniformly converges to zero due to the continuity of f. Now, uniform convergence and integration are commutative limits, and thus

$$
\begin{aligned}
\int_0^x f(t)\,dt &= \lim_{m\to\infty}\int_0^x \mathbf{Q}_m(t)\,dt \\
&= \lim_{m\to\infty}\left(2^{km}\triangle^k\mathbf{P}_m(x) - 2^{km}\,\triangle^k\mathbf{p}_0^m\right) \\
&= \tilde{g}(x) - \tilde{g}(0),
\end{aligned}
$$

i.e., $\tilde{g}$ is continuously differentiable with $\frac{d}{dx}\,\tilde{g} = f$. It is exactly this conclusion that makes it necessary to require convergence to happen with respect to the $\|\cdot\|_\infty$ norm.

We assumed that from $2^{km}\triangle^k\mathbf{P}_m \to \tilde{g} \in C^0$ it follows that $\mathbf{P}_m$ converges to some $g \in C^k$ with $g^{(k)} = \tilde{g}$ and showed actually $g \in C^{k+1}$ since $g^{(k+1)} = \frac{d}{dx}\,\tilde{g} = f \in C^0$ which concludes the induction. □

Now we have the tool-kit ready to prove the following sufficient convergence criterion.

Theorem 5 *Let $[\mathbf{P}_m]$ be a sequence of polygons generated by the iterative application of an interpolatory refinement scheme. If*

$$
\sum_{m=0}^{\infty} \|2^{km}\triangle^{k+l}\mathbf{P}_m\|_\infty < \infty \tag{8}
$$

for some $l \in \mathbb{N}$, then the sequence $[\mathbf{P}_m]$ uniformly converges to a k-times continuously differentiable limit curve.

Proof For $l > 1$, we apply Lemma 3 $(l-1)$-times to obtain the same statement about the rate of contraction for the scalar sequence $[2^{km}\|\triangle^{k+1}\mathbf{P}_m\|_\infty]_m$. By Lemma 2, these values bound the maximum distances between successive difference polygons $2^{km}\triangle^k\mathbf{P}_m$. Thus the sequence $[2^{km}\triangle^k\mathbf{P}_m]_m$ is a Cauchy-sequence and, since every element of this sequence is a continuous function, it is well known that it converges to a continuous limit curve. Finally, Lemma 4 concludes the proof. □

Applying this theorem allows to reduce the convergence analysis of interpolatory refinement schemes to the analysis of the rate of contraction of some arbitrarily high forward differences. This sufficient condition can be tested without explicitly referring to the special equidistant parameterization with step-width $h_{m,k}$, i.e., we just have to prove the convergence of the scalar series (8) without caring about how the $\mathbf{P}_m$ were generated. The theorem then guarantees that there exists a regular reparameterization (based on the $h_{m,k}$) such that the coordinate functions of the limit curve $\mathbf{P}_\infty$ are C^k on some compact interval.

Theorem 5 somewhat generalizes the well-known convergence criteria of [Dyn91] in the case of interpolatory subdivision schemes. The sufficient conditions in [Dyn91] require the existence of a constant factor $\rho < 1$ by which the maximum $(k+1)$th forward difference has to decrease after a fixed number of subdivision steps. This however would imply that (8) can be bounded by a convergent geometric series. Hence, Theorem 5 is more general since it allows any kind of converging series and does not require exponential convergence (which, however, is the only kind of convergence *linear* operators are able to produce). Nevertheless, the more important generalization is that Theorem 5 does not require the subdivision scheme to be stationary, compactly supported or even linear.

We can state a very similar convergence criterion in the frequency domain if we look at the discrete difference spectra $\triangle^{k+l}\widehat{\mathbf{P}}_m$ (cf. (6)) instead of the original difference polygons $\triangle^{k+l}\mathbf{P}_m$. Notice that the application of the DFT implies a periodic structure of the vector components. Thus, while Theorem 5 holds for open and closed polygons as well, the next corollary is only suitable for the refinement of closed polygons.

Corollary 6 *Let $[\mathbf{P}_m]$ be a sequence of closed polygons generated by the iterative application of an interpolatory refinement scheme. If the difference spectra satisfy*

$$\sum_{m=0}^{\infty} \|2^{(k-1)m} \triangle^{k+l}\widehat{\mathbf{P}}_m\|_1 < \infty$$

for some $l \in \mathbb{N}$, then the original sequence $[\mathbf{P}_m]$ converges to a k-times continuously differentiable limit curve.

Proof Use the inequality (5) and notice that the number of vertices in the polygons $\triangle^k\mathbf{P}_m$ increases like $O(2^m)$ as m tends to ∞. $\square$

It is important to notice that corollary 6 allows to check the convergence of a subdivision scheme by estimating the $\|\cdot\|_1$-norm (i.e., the average function value) of the difference spectrum. This will turn out to lead to sharper convergence criteria than those based on testing the $\|\cdot\|_\infty$-norm.

4 Implicit refinement schemes

According to (2) an implicit refinement scheme defines the new vertices $\mathbf{p}_{2i+1}^{m+1}$ of the refined polygon $\mathbf{P}_{m+1}$ by

$$\sum_{k \in \mathbb{Z}} \beta_k \, \mathbf{p}_{2i+1+k}^{m+1} \;=\; 0, \qquad \forall\, i \tag{9}$$

under the condition that $\mathbf{p}_{2i}^{m+1} = \mathbf{p}_i^m$. The coefficients β_k build a finite sequence which in most practical cases can be assumed symmetric, i.e., $\beta_k = \beta_{-k}$. The implicit definition (9) is in contrast to common approaches where the new vertices are computed by *explicit* rules (cf. (1)).

By separating the unknown $\mathbf{p}_{2i+1}^{m+1}$ from the known $\mathbf{p}_{2i}^{m+1} = \mathbf{p}_i^m$ in (9) we obtain a linear system

$$\mathcal{B}\,[\mathbf{p}_{2i+1}^{m+1}]_i \;=\; \mathcal{C}\,[\mathbf{p}_{2i}^{m+1}]_i \tag{10}$$

which has to be solved for the $\mathbf{p}_{2i+1}^{m+1}$. Both, $\mathcal{B}$ and $\mathcal{C}$ are banded convolution operators. Applying the z-transform to the system, yields the polynomial identity

$$\Big(\sum_k \beta_{2k}\, z^{-2k}\Big)\Big(\sum_k \mathbf{p}_{2k+1}^{m+1}\, z^{2k+1}\Big) \;=\; -\Big(\sum_k \beta_{2k-1}\, z^{-(2k-1)}\Big)\Big(\sum_k \mathbf{p}_k^m\, z^{2k}\Big)$$

or equivalently

$$\begin{aligned}\sum_k \mathbf{p}_k^{m+1}\, z^k \;&=\; \left(1 - \frac{\sum_k \beta_{2k-1}\, z^{-(2k-1)}}{\sum_k \beta_{2k}\, z^{-2k}}\right) \sum_k \mathbf{p}_k^m\, z^{2k}\\ &=\; \left(\frac{\sum_k (-1)^k\, \beta_k\, z^{-k}}{\sum_k \beta_{2k}\, z^{-2k}}\right) \sum_k \mathbf{p}_k^m\, z^{2k}\\ &=:\; H(z) \sum_k \mathbf{p}_k^m\, z^{2k}.\end{aligned} \tag{11}$$

Hence, the implicit refinement schemes correspond to discrete filters with a special form of rational transfer function $H(z)$ which due to the construction guarantees cardinal interpolation (half-band property). For $\beta_{2k} = \delta_{k,0}$, the transfer function is polynomial and the filter has a finite impulse response (FIR).

In [Kob96] interpolatory refinement schemes are derived from a variational set-up, i.e., a quadratic energy functional is defined and the new vertices $\mathbf{p}_{2i+1}^{m+1}$ are chosen such that $\mathbf{P}_{m+1}$ minimizes this functional. The resulting refinement schemes are implicit schemes with (9) playing the role of the discrete *Euler-Lagrange equation* of the optimization problem.

The use of the particular energy functionals $E(\mathbf{P}_{m+1}) := \|\triangle^r\, \mathbf{P}_{m+1}\|_2$ leads to an Euler-Lagrange equation of the form

$$\triangle^{2r}\mathbf{p}_{2i+1-r}^{m+1} \;=\; 0, \qquad \forall\, i. \tag{12}$$

Hence, the coefficients β_k in this case are the binomial coefficients $\beta_k = (-1)^k \binom{2r}{r+k}$ and the corresponding transfer function takes the simple form

$$H(z) = \frac{2\,(z+1)^{2r}}{(z+1)^{2r} + (z-1)^{2r}} \tag{13}$$

with the frequency spectrum

$$|H(j\omega)| \;=\; \frac{2\,\cos(\frac{\omega}{2})^{2r}}{\cos(\frac{\omega}{2})^{2r} + \sin(\frac{\omega}{2})^{2r}}.$$

Thus, the special implicit refinement schemes given by (12) are exactly the $2r$-th order Butterworth half-band filters [OS89], [HV93] whose transfer function is maximally flat in the sense that a maximum number of derivatives vanishes at $\omega = 0$. The connection between the implicit refinement schemes (12) and the Butterworth filters has been pointed out to me by A. Cohen and an unknown referee.

5 Analysis of interpolatory refinement schemes

In this section, we will use the discrete Fourier transform to reformulate the action of refinement operators in order to derive simple convergence criteria. The use of the discrete Fourier transform requires that the transformed sequences be periodic. Therefore we focus on the refinement of closed polygons in this section. Since we want to apply Theorem 5, we restrict ourselves to interpolatory schemes, more precisely we assume the refinement operator to belong to the class of implicit refinement schemes. The case of non-interpolatory stationary schemes is investigated in Section 6.

To understand the action of a refinement operator in the discrete frequency domain, we look at the interpolatory refinement in detail. Starting with the given polygon $\mathbf{P}_m = [\mathbf{p}_0^m, \ldots, \mathbf{p}_{n-1}^m]$ and its Fourier transform $\widehat{\mathbf{P}}_m = [\hat{\mathbf{p}}_0^m, \ldots, \hat{\mathbf{p}}_{n-1}^m]$, we first define the vector $\mathbf{P}'_m = [\mathbf{p}_0^m, 0, \ldots, \mathbf{p}_{n-1}^m, 0]$ which coincides with $\mathbf{P}_{m+1}$ in the even components due to the interpolation property $\mathbf{p}_{2i}^{m+1} = \mathbf{p}_i^m$. In the frequency domain the corresponding spectrum is

$$\widehat{\mathbf{P}}'_m \;=\; \Big[\hat{\mathbf{p}}_0^m, \ldots, \hat{\mathbf{p}}_{n-1}^m, \hat{\mathbf{p}}_0^m, \ldots, \hat{\mathbf{p}}_{n-1}^m\Big].$$

The odd components of the refined polygon $\mathbf{P}_{m+1}$ are computed from $\mathbf{P}_m$ by applying a circulant matrix $\mathcal{A}$ which can be factorized into $\mathcal{A} = \mathcal{B}^{-1}\,\mathcal{C}$ with both $\mathcal{B}$ and $\mathcal{C}$ having a bounded bandwidth (cf. the solution of (10)). The vector $\mathbf{P}''_m := \mathbf{P}_{m+1} - \mathbf{P}'_m$ is obtained from $\mathcal{A}\,\mathbf{P}_m$ by the insertion of zeros for the even components. Its Fourier transform is

$$\widehat{\mathbf{P}}''_m \;=\; \Big[\omega_{2n}^i\,\mu_i\,\hat{\mathbf{p}}_i^m\Big]_{i=0}^{2n-1}$$

where $\mu_0, \ldots, \mu_{n-1}$ are the diagonal elements of $\widehat{\mathcal{A}}$ and the index i of μ_i and $\hat{\mathbf{p}}_i^m$ has to be taken *modulo* n. Finally, the Fourier transform of $\mathbf{P}_{m+1} = \mathbf{P}'_{m+1} + \mathbf{P}''_{m+1}$ is

$$\widehat{\mathbf{P}}_{m+1} \;=\; \Big[(1 + \omega_{2n}^i\,\mu_i)\,\hat{\mathbf{p}}_i^m\Big]_{i=0}^{2n-1} \;=:\; \Big[\lambda_i\,\hat{\mathbf{p}}_i^m\Big]_{i=0}^{2n-1}.$$

Due to the factorization $\mathcal{A} = \mathcal{B}^{-1}\,\mathcal{C}$ with banded circulant matrices $\mathcal{B}$ and $\mathcal{C}$, there exists a (trigonometric rational) transfer function $\lambda : \mathbb{R} \to \mathbb{C}$ (cf. $H(j\omega)$ in Section 4) which does not depend on n and for which

$$\lambda_i \;=\; \lambda\Big(\frac{i}{2n}\Big) \;=\; 1+\mu\Big(\frac{i}{2n}\Big) \qquad \text{and} \qquad \lambda(x+1) \;=\; \lambda(x).$$

Since the refinement scheme is assumed to be interpolatory having only real-valued coefficients which are symmetric (asymmetric schemes in a uniform setting make no sense from the geometric point of view), we further find from the special structure of $H(z)$ in (11) that μ is real-valued with several symmetries

$$\mu(x) \;=\; -\mu(\frac{1}{2}-x) = -\mu(\frac{1}{2}+x) = \mu(1-x)$$

and $\mu(\frac{1}{4}) = 0$ (half-band property). The affine invariance of the refinement scheme is equivalent to constant precision and thus can be guaranteed by $\mu(0) = 1$. We call a refinement scheme satisfying all these assumptions *geometrically meaningful*.

Example Consider the minimally supported refinement scheme which reproduces the piecewise linear interpolant through the initial vertices $\mathbf{p}_i^0$. The new vertices are implicitly given by the conditions

$$\mathbf{p}_{2i}^{m+1} - 2\,\mathbf{p}_{2i+1}^{m+1} + \mathbf{p}_{2i+2}^{m+1} \;=\; 0, \qquad i = 0,\ldots,n\,2^m-1$$

and the convolution operator by which the new vertices with odd index are computed is $\mathcal{A} = \mathrm{circ}[\frac{1}{2},\frac{1}{2},0,\ldots,0]$. The corresponding transfer function is

$$\lambda(x) \;=\; 1+e^{2\pi j x}\left(\frac{1}{2}+\frac{1}{2}\,e^{-4\,\pi\,j\,x}\right) \;=\; 1+\cos(2\,\pi\,x).$$

By applying a refinement scheme with transfer function $\lambda(x)$ iteratively to the given polygon $\mathbf{P}_m = [\mathbf{p}_0^m,\ldots,\mathbf{p}_{n-1}^m]$ we obtain in the frequency domain

$$\widehat{\mathbf{P}}_{m+r} \;=\; \Big[\,\prod_{s=1}^{r}\lambda\Big(\frac{i}{n\,2^s}\Big)\;\hat{\mathbf{p}}_i^m\,\Big]_{i=0}^{n\,2^r-1} \tag{14}$$

where the index i of $\hat{\mathbf{p}}_i^m$ again is taken *modulo* n. Considering the difference polygon $\triangle^k\mathbf{P}_m$, it follows from (6) that

$$\triangle^k\widehat{\mathbf{P}}_{m+r} \;=\; \Big[\,\prod_{s=1}^{r}\frac{(\omega_{n\,2^s}^{-i}-1)^k}{(\omega_{n\,2^{s-1}}^{-i}-1)^k}\;\lambda(\frac{i}{n\,2^s}) \quad (\omega_n^{-i}-1)^k\;\hat{\mathbf{p}}_i^m\,\Big]_{i=0}^{n\,2^r-1}. \tag{15}$$

Thus, the *derived* transfer function $\lambda_{(k)}(x)$ which maps $\triangle^k\widehat{\mathbf{P}}_m$ directly to $\triangle^k\widehat{\mathbf{P}}_{m+1}$ can be written as

$$\lambda_{(k)}(x) \;:=\; \frac{(e^{j\,2\,\pi\,x}-1)^k}{(e^{j\,4\,\pi\,x}-1)^k}\,\lambda(x) \;=\; \frac{\lambda(x)}{(e^{j\,2\,\pi\,x}+1)^k}.$$

This derived function has a singularity at $x = \frac{1}{2}$ if λ has a zero of order less than k. Hence, it does not make sense to consider difference spectra of higher order than the root of λ at $x = \frac{1}{2}$. The modulus of the possibly complex valued function $\lambda_{(k)}(x)$ is

$$|\lambda_{(k)}(x)| \;=\; \frac{|\lambda(x)|}{2^k\,|\cos(\pi\,x)|^k}. \tag{16}$$

Now, using the convergence criterion of Corollary 6, we are able to prove the following

Theorem 7 *Let $\lambda(x)$ be the transfer function of an interpolatory refinement scheme, with a root of order $(k+l)$ at $\frac{1}{2}$ for some $l \in \mathbb{N}$. Then the scheme produces k-times continuously differentiable limit curves if, for some $r \in \mathbb{N}$ and $q < 2^{-k}$,*

$$\max_{x\in[0,1]} \;\; \frac{1}{2^r}\sum_{h=0}^{2^r-1}\prod_{s=1}^{r}\left|\lambda_{(k+l)}\left(\frac{x+h}{2^s}\right)\right| \;\le\; q^r.$$

Proof Let $\triangle^{k+l}\widehat{\mathbf{P}}_m = [\,(\omega_n^{-i}-1)^{k+l}\,\hat{\mathbf{p}}_i^m\,]_{i=0}^{n-1}$. Then

$$\begin{aligned}
\|\triangle^{k+l}\widehat{\mathbf{P}}_{m+r}\|_1 \;&=\; \sum_{i=0}^{n\,2^r-1}\prod_{s=1}^{r}\left|\lambda_{(k+l)}\Big(\frac{i}{n\,2^s}\Big)\right|\,|(\omega_n^{-i}-1)^{k+l}\,\hat{\mathbf{p}}_i^m|\\
&\le\; \|\triangle^{k+l}\widehat{\mathbf{P}}_m\|_1 \;\max_{i\in[0,n-1]}\sum_{h=0}^{2^r-1}\prod_{s=1}^{r}\left|\lambda_{(k+l)}\Big(\frac{i+n\,h}{n\,2^s}\Big)\right|\\
&\le\; \|\triangle^{k+l}\widehat{\mathbf{P}}_m\|_1 \;\max_{x\in[0,1]}\sum_{h=0}^{2^r-1}\prod_{s=1}^{r}\left|\lambda_{(k+l)}\Big(\frac{x+h}{2^s}\Big)\right|
\end{aligned}$$

where we exploit the periodicity of the $(\omega_n^{-i}-1)^{k+l}\,\hat{\mathbf{p}}_i^m$. From the assumed condition it follows

$$\|\triangle^{k+l}\widehat{\mathbf{P}}_m\|_1 = O\big((2q)^m\big),$$

and this, by Corollary 6, is sufficient for the convergence to a C^k function since $2q < 2^{1-k}$.
□

In this theorem, the parameter r counts how many refinement steps are combined for the estimation of the rate of convergence. Due to the averaging behavior $(\frac{1}{n}\sum\cdots)$ of the estimated term in the sufficient condition, the evaluation can be performed in a numerically stable way which is not the case for methods based on the continuous Fourier transform where the supremum of functions of the form $\prod_s |\lambda_{(k+l)}(x\,2^s)|$ has to be estimated (cf. Fig. 2). Since, from the symmetry of $\mu(x)$, it follows that $\lambda_{(k+l)}(x) = \lambda_{(k+l)}(1-x)$, it suffices to estimate the maximum for $x \in [0, \frac{1}{2}]$.

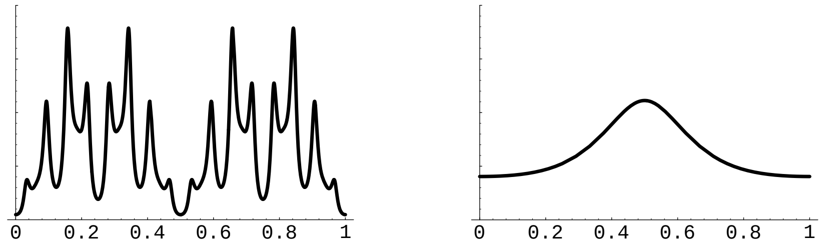

Figure 2: Examples for the typical oscillations of functions of the form $\prod_s |\lambda_{(k+l)}(x\, 2^s)|$ (left) and $\frac{1}{2^r} \sum_h \prod_s |\lambda_{(k+l)}((x+h)\, 2^{-s})|$ (right). Here $r = 4$ for the transfer function of the example in Section 7.2.

If we set $r = 1$ in Theorem 7, we get a weaker but more intuitive sufficient condition for the convergence. Suppose μ is (piecewise) continuous and $|\mu(x)| \le 1$. This is satisfied if the refinement matrix $\mathcal{A}$ has no negative eigenvalues. Together with the symmetry of $\mathcal{A}$, this means that the refinement operator has the *linear phase property* [Rio92]. Then, $\lambda(x) \ge 0$ and the sufficient condition from Theorem 7 is equivalent to

$$\frac{\sin(\pi\, \dfrac{x}{2})^{k+l} + \cos(\pi\, \dfrac{x}{2})^{k+l} + \mu(\dfrac{x}{2}) \left(\sin(\pi\, \dfrac{x}{2})^{k+l} - \cos(\pi\, \dfrac{x}{2})^{k+l}\right)}{\sin(\pi\, x)^{k+l}} < \frac{1}{2^{k-1}}, \qquad (17)$$

for $x \in [0, \frac{1}{2}]$ as follows from the symmetry of μ and (16). For $x = \frac{1}{2}$, this is satisfied iff $l \ge k+1$, i.e., one has to analyze at least the behavior of the $(2k+1)$-th difference polygons if one wants to prove $\mathbf{P}_\infty \in C^k$ by using Theorem 7 with $r = 1$.

For $x \to 0$ one obtains

$$\lim_{x \to 0+} \frac{\sin(\pi\, \dfrac{x}{2})^{k+l} + \cos(\pi\, \dfrac{x}{2})^{k+l} + \mu(\dfrac{x}{2}) \left(\sin(\pi\, \dfrac{x}{2})^{k+l} - \cos(\pi\, \dfrac{x}{2})^{k+l}\right)}{2^{k+l}\, \sin(\pi\, \dfrac{x}{2})^{k+l}\, \cos(\pi\, \dfrac{x}{2})^{k+l}}$$

$$= \lim_{x \to 0+} \frac{1}{2^{k+l}} \left(\frac{1 + \mu(\frac{x}{2})}{\cos(\pi\, \frac{x}{2})^{k+l}} + \frac{1 - \mu(\frac{x}{2})}{\sin(\pi\, \frac{x}{2})^{k+l}} \right).$$

Let $\mu(x)$ be sufficiently differentiable in some interval $[0, \varepsilon]$. Then, by the symmetry of μ and since $\lambda(x)$ has a root of order $(k+l)$ at $x = \frac{1}{2}$, we have $\mu(0) = 1$ and $\mu'(0) = \cdots = \mu^{(k+l-1)}(0) = 0$. Hence, one can apply de l'Hôpital's rule to the second term:

$$\lim_{x \to 0+} \frac{1 - \mu(\frac{x}{2})}{\sin(\pi\, \frac{x}{2})^{k+l}} \;=\; \cdots \;=\; \lim_{x \to 0+} \frac{-2^{-(k+l)}\, \mu^{(k+l)}(\frac{x}{2})}{(k+l)!\, (\frac{\pi}{2})^{k+l} \cos(\pi\, \frac{x}{2})^{k+l} + O(\sin(\pi\, \frac{x}{2}))}$$

$$= \frac{-\mu^{(k+l)}(0)}{(k+l)!\, \pi^{k+l}}.$$

Thus the condition (17) is surely satisfied for $x \to 0$ if $\lambda(x)$ even has a root of order $(k+l+1)$ at $x = \frac{1}{2}$. For $0 < x < \frac{1}{2}$, the expressions in (17) have no singularities and therefore we obtain the following

Corollary 8 *Let $\lambda(x) = 1 + \mu(x)$ be the transfer function of a geometrically meaningful interpolatory refinement scheme with a root of order $(k+l+1)$ at $x = \frac{1}{2}$ for some $l \geq k+1$. Then the scheme produces C^k-curves if for all $x \in (0, \frac{1}{4})$*

$$\frac{2^{1-k}\,\sin(2\,\pi\,x)^{k+l} - \sin(\pi\,x)^{k+l} - \cos(\pi\,x)^{k+l}}{\sin(\pi\,x)^{k+l} - \cos(\pi\,x)^{k+l}} \;<\; \mu(x) \;\leq\; 1.$$

We call the area where the graph of $\mu(x)$ is restricted to lie, the C^k-*corridor* (cf. Fig. 3). Corollary 8 can be used in two directions. Given a particular refinement scheme, we can compute the corresponding transfer function $\lambda(x) = 1 + \mu(x)$ and test whether $\mu(x)$ lies within some C^k-corridor (*analysis*). On the other hand, we can *construct* new schemes in a simple and systematic way by choosing a transfer function from some C^k-corridor and then derive the corresponding refinement scheme.

6 Analysis of stationary schemes

Since Theorem 5 is known to hold for non-interpolatory stationary schemes as well [Dyn91], we can apply Theorem 7 to this class. In this case the transfer function λ (which has to be evaluated at the points $x = \frac{i}{2n}$ if a polygon $\mathbf{P} = [\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}]$ is to be subdivided) is the sum of two trigonometric polynomials μ_0 and μ_1, i.e.,

$$\lambda(x) \;=\; \mu_0(x) + \mu_1(x) \;=\; \sum_i \alpha_i\, e^{-2\,\pi\,j\,x\,i}$$

with

$$\mu_0(x) \;=\; \sum_i \alpha_{2i}\, e^{-4\,\pi\,j\,x\,i} \qquad \text{and} \qquad \mu_1(x) \;=\; e^{-2\,\pi\,j\,x} \sum_i \alpha_{2i+1}\, e^{-4\,\pi\,j\,x\,i},$$

where the coefficients α_i constitute the subdivision mask of (1). Comparing this to the continuous Fourier method explained in the introduction, we see that

$$\lambda(x) \;=\; 2\,\alpha(2\,\pi\,x)$$

and

$$\lambda_{(k+l)}(x) \;=\; \frac{2}{(e^{j\,2\,\pi\,x}+1)^{k+l}}\;\alpha(2\,\pi\,x) \;=:\; 2^{1-k-l}\,\mathcal{L}(2\,\pi\,x).$$

A well-known sufficient criterion for the differentiability of the limit curve of convergent subdivision schemes is [Dau92, Lemma 7.1.2]

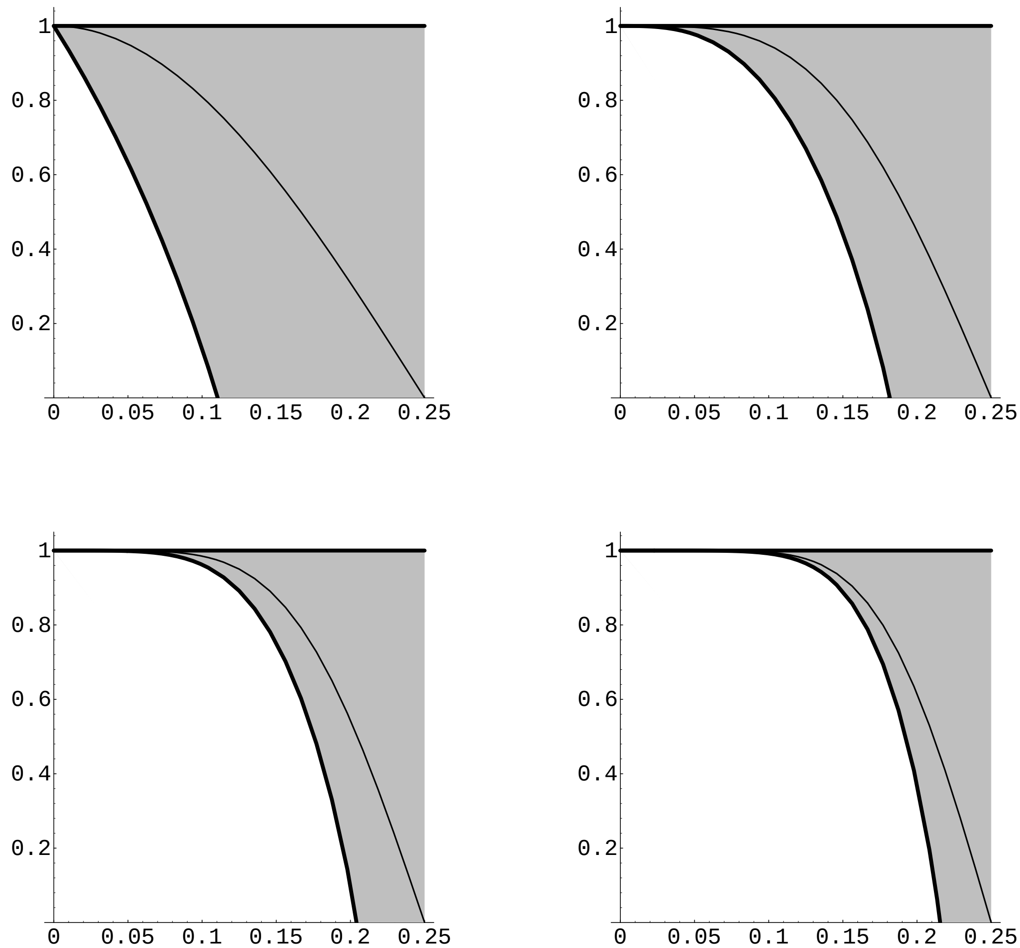

Figure 3: C^k-corridors (grey) for $k = 0, 1, 2, 3$ with $l = k + 1$ and examples for typical functions $\mu(x)$ corresponding to refinement schemes that generate C^k limit curves.

$$\sup_{y \in \mathbb{R}} \left| \prod_{s=1}^{r} \mathcal{L}(2^{-s}\, y) \right| \;<\; 2^{r(l-1)} \quad \Longrightarrow \quad \Phi \in C^k. \tag{18}$$

If this condition is satisfied, it follows

$$\max_{x\in[0,1]} \quad \frac{1}{2^r} \sum_{h=0}^{2^r-1} \prod_{s=1}^{r} \left| \lambda_{(k+l)} \left(\frac{x+h}{2^s} \right) \right| \;<\; \frac{1}{2^r} \sum_{h=0}^{2^r-1} \; 2^{r(1-k-l)+r(l-1)} \;=\; 2^{-rk},$$

which proves that the convergence criterion 7 for the same r is more general. The improvement stems from the discrete Fourier transform which allows the use of the $\|\cdot\|_1$-norm instead of the $\|\cdot\|_\infty$-norm as motivated by (5). This yields a much sharper estimate since the average function value of the transfer function is usually much smaller than its maximum. In Section 7.3, numerical results are given which allow to compare both criteria.

7 Some Applications

To demonstrate the use of the discrete Fourier approach, we give some examples of how the discrete formalism can be applied to the analysis and construction of refinement schemes.

7.1 Implicit refinement schemes

Consider a particular subset of the implicit refinement schemes (2):

$$\mathbf{p}_{2i+1}^{m+1} + \beta\,(\mathbf{p}_{2i}^{m+1} + \mathbf{p}_{2i+2}^{m+1}) - \Big(\frac{1}{2} + \alpha + \beta\Big)\,(\mathbf{p}_{2i-1}^{m+1} + \mathbf{p}_{2i+3}^{m+1}) + \alpha\,(\mathbf{p}_{2i-2}^{m+1} + \mathbf{p}_{2i+4}^{m+1}) = 0$$

with $-1 < \alpha + \beta < 0$. The restrictions on the coefficients α and β are introduced in order to guarantee symmetry, solvability and affine invariance of the scheme [Kob95]. The transfer function

$$\lambda(x) = 1 + \frac{2\,\beta\,\cos(2\,\pi\,x) + 2\,\alpha\,\cos(6\,\pi\,x)}{(1 + 2\,\alpha + 2\,\beta)\,\cos(4\,\pi\,x) - 1}$$

can easily be read off from the coefficients of the implicit scheme without further computations (cf. (11)). This becomes obvious if one looks at the Fourier transform of the matrix $\mathcal{A} = \mathcal{B}^{-1}\,\mathcal{C}$ by which the new vertices are computed from the old.

If we want use Corollary 8 to determine which values α and β do represent refinement schemes that generate at least C^1 limit curves, then the corresponding transfer function $\lambda(x) = 1 + \mu(x)$ has to have a root of order 4 at $x = \frac{1}{2}$. Comparing the coefficients of the Taylor expansion of $\lambda(x)$ shows that this is achieved if

$$3\,\beta = 5\,\alpha - 2.$$

Hence, we have a family of implicit refinement schemes

$$\mathbf{p}_{2i+1}^{m+1} + \frac{5\,\alpha - 2}{3}\,(\mathbf{p}_{2i}^{m+1} + \mathbf{p}_{2i+2}^{m+1}) - \frac{16\,\alpha - 1}{6}\,(\mathbf{p}_{2i-1}^{m+1} + \mathbf{p}_{2i+3}^{m+1}) + \alpha\,(\mathbf{p}_{2i-2}^{m+1} + \mathbf{p}_{2i+4}^{m+1}) = 0$$

with one free parameter α and the corresponding transfer function

$$\lambda(x) = 1 + \frac{(10\,\alpha - 4)\,\cos(2\,\pi\,x) + 6\,\alpha\,\cos(6\,\pi\,x)}{(16\,\alpha - 1)\,\cos(4\,\pi\,x) - 3}. \tag{19}$$

Using Corollary 8 it is easy to prove numerically that the limit curve is C^1 if $\alpha = -\frac{1}{20}$ or $\alpha = \frac{1}{5}$. Since $\mu(x)$ is monotonic with respect to α, it follows that the limit curve is C^1 for all $\alpha \in [-\frac{1}{20}, \frac{1}{5}]$. Moreover, it is possible to verify, by applying Theorem 7, that the limit is C^2 for $\alpha \in [-\frac{1}{20}, 0]$ and C^3 at least for $\alpha \in [-\frac{1}{20}, -0.0373]$. In the case $\alpha = -\frac{1}{20}$, the transfer function has a root of order 6 and we even get C^4-curves. Notice that $\alpha = \frac{1}{16}$ reproduces the 4-point scheme of [DGL87].

To prove these higher regularities, one has to combine several refinement steps. Best results in the convergence analysis by Theorem 7 are achieved if l is taken to its maximum, i.e., $(k + l) = 4$ for $\alpha \neq -\frac{1}{20}$ and $(k + l) = 6$ for $\alpha = -\frac{1}{20}$.

7.2 Variational Schemes

As explained in Section 4, the implicit refinement schemes can be considered as discrete Euler-Lagrange equations characterizing the solution of some optimization problem.

Let us consider a particular quadratic energy functional which measures the *strain* of a polygon $\mathbf{P}_{m+1}$ by

$$\mathbf{E}(\mathbf{P}_{m+1}) \ := \ \| \triangle^3 \mathbf{P}_{m+1} \|_2 \ = \ \sum_i \|\triangle^3 \mathbf{p}_i^{m+1}\|_2^2.$$

The vertices with even index are fixed due to the interpolation condition while the vertices with odd index are the free variables of the optimization problem. The discrete Euler-Lagrange-equation for this problem is[2]

$$\triangle^6 \mathbf{p}_{2i}^{m+1} = 0, \qquad i = 0, \ldots, n-1.$$

Hence, the new vertices of the refined polygon are the solution of

$$\mathrm{circ}[20, 6, 0, \ldots, 0, 6] \ [\mathbf{p}_{2i+1}^{m+1}]_{i=0}^{n-1} \ = \ \mathrm{circ}[15, 15, 1, 0, \ldots, 0, 1] \ [\mathbf{p}_i^m]_{i=0}^{n-1}.$$

The diagonal elements of the Fourier transformed convolution matrices $\mathcal{B}$ and $\mathcal{C}$ (cf. (10)) are

$$\mu_i \ = \ 20 + 12 \cos\Big(\frac{2\,\pi\, i}{n}\Big) \qquad \text{and} \qquad \nu_i \ = \ 15 + 15\, \omega_n^{-i} + \omega_n^{-2i} + \omega_n^i$$

respectively. The resulting transfer function $\lambda(x)$ corresponding to the convolution matrix $\mathcal{A} = \mathcal{B}^{-1}\mathcal{C}$ is the trigonometric rational polynomial

$$\lambda\Big(\frac{i}{2n}\Big) \ = \ 1 + \omega_{2\,n}^i \frac{\nu_i}{\mu_i} \ = \ 1 + \frac{15 \cos(\frac{2\,\pi\, i}{2n}) + \cos(\frac{6\,\pi\, i}{2n})}{10 + 6 \cos(\frac{4\,\pi\, i}{2n})}.$$

The derived transfer function $\lambda_{(6)}(x)$ takes the simple form

$$\begin{aligned} |\lambda_{(6)}(x)| \ &= \ \frac{1}{64 \cos(\pi\, x)^6} \left| 1 + \frac{15 \cos(2\,\pi\, x) + \cos(6\,\pi\, x)}{10 + 6 \cos(4\,\pi\, x)} \right| \\ &= \ \frac{1}{20 + 12 \cos(4\,\pi\, x)} \end{aligned}$$

and since

$$|\lambda_{(6)}(x)| \ \le \ \Big|\lambda_{(6)}\Big(\frac{1}{4}\Big)\Big| \ = \ \frac{1}{8} \ < \ \frac{1}{4},$$

[2] Compare this to the non-discrete case, where $f^{(6)}(x) \equiv 0$ is the Euler-Lagrange-equation for the minimization of $\int [f^{(3)}(x)]^2\, dx$.

it follows immediately from Theorem 7 with $r = 1$ that $\mathbf{P}_\infty \in C^2$. Collecting two refinement steps, it can be shown that $\mathbf{P}_\infty \in C^3$ and, for $r = 11$, we even obtain $\mathbf{P}_\infty \in C^4$. Notice that this particular refinement scheme belongs to the class of schemes presented in the last subsection with $\alpha = -\frac{1}{20}$.

More generally, if we minimize

$$\mathbf{E}(\mathbf{P}_{m+1}) \;:=\; \| \triangle^r \mathbf{P}_{m+1} \|_2 \;=\; \sum_i \|\triangle^r \mathbf{p}_i^{m+1}\|_2^2,$$

we obtain the corresponding Euler-Lagrange equation

$$\triangle^{2r} \mathbf{p}_{2i+1-r}^{m+1} = 0, \qquad i = 0, \ldots, n-1.$$

Applying the z-transform as in (13) and writing the resulting transfer function as a rational polynomial $H(z) = H(e^{-2\pi j x})$, we get

$$|\lambda_{2r}(x)| \;=\; \frac{|H(z)|}{(z+1)^{2r}} \;=\; \frac{2^{1-2r}}{\cos(2\,\pi\,x)^{2r} + \sin(2\,\pi\,x)^{2r}}$$

and since

$$|\lambda_{2r}(x)| \;\leq\; \left|\lambda_{2r}\left(\frac{1}{4}\right)\right| \;=\; 2^{-r} < 2^{1-r}$$

we immediately see that the variational scheme minimizing $\| \triangle^r \mathbf{P}_{m+1} \|_2$ generates at least C^{r-1} limit curves. In [Kob96] it is shown that the combination of two refinement steps yields a sharper bound: The iterative minimization of $\| \triangle^r \mathbf{P}_{m+1} \|_2$ generates at least C^r limit curves (and higher continuity is to be expected if more refinement steps are combined).

7.3 Stationary subdivision

In the case of stationary subdivision schemes, the continuous and the discrete Fourier methods can be compared by looking at the number r of steps that have to be combined in order to prove a certain differentiability of the limit curves. We applied both criteria, Theorem 7 and (18), to the Lagrange schemes [DD89] of order $k = 2, 3, 4$. The results are given in Table 1. The differentiability of the limit curves has to be *strictly less* than the indicated upper bounds.

7.4 C^∞-Interpolants

The last example shows how to construct a refinement scheme which produces interpolating C^∞-curves. Looking at the transfer function

order k	steps r	diff'ty (cont.)	diff'ty (discr.)
2	1	1.415	2.0
2	2	1.651	2.0
2	8	1.667	2.0
3	1	1.678	2.541
3	2	2.212	2.746
3	8	2.247	2.804
4	1	1.871	2.83
4	2	2.729	3.39
4	8	2.785	3.493
4	15	3.072	3.52
4	20	3.2	3.528

Table 1: Upper bounds for the differentiability of the Lagrange interpolatory subdivision schemes. In the third column are the results from the classical continuous Fourier method and the fourth column contains the results obtain by applying the discrete Fourier technique. The improvement stems from the use of the $\|\cdot\|_1$-norm instead of the $\|\cdot\|_\infty$-norm.

$$\lambda_\infty(x) \;=\; 1+\mu_\infty(x) \;=\; \begin{cases} 2 & 0 \le x < \frac{1}{4}, \quad \frac{3}{4} < x < 1 \\ 1 & x = \frac{1}{4}, \quad x = \frac{3}{4} \\ 0 & \frac{1}{4} < x < \frac{3}{4} \end{cases}$$

it is easy to verify that the corresponding refinement scheme is geometrically meaningful and produces C^∞-curves since $\mu_\infty(x)$ passes through the C^k-corridor for every $k \in \mathbb{N}$.

In fact, since $\lambda_\infty(x)$ vanishes for $\frac{1}{4} < x < \frac{3}{4}$, no higher frequencies are introduced through the subdivision operator. Hence, this refinement scheme actually computes the minimally band limited interpolant through the given vertices $\mathbf{P}_0 = [\mathbf{p}_0^0, \ldots, \mathbf{p}_{n-1}^0]$ which is, assuming a parameterization $\mathbf{P}_\infty(i) = \mathbf{p}_i^0$, uniquely defined by a linear combination of the integer shifts of the cardinal basis function [GY83]

$$\Psi(x) \;=\; \sum_{i\,\in\,\mathbb{Z}} \frac{\sin(\pi\,(x - i\,n))}{\pi\,(x - i\,n)}.$$

This refinement scheme can be understood as the *limit scheme* of the class of Lagrange-schemes (as $k \to \infty$). Although the scheme is global, its computational complexity is not very high. The polygon $\mathbf{P}_m$ on the m-th refinement level can be obtained in the following way: First compute the Fourier transform $\widehat{\mathbf{P}}_0$ of the given polygon $\mathbf{P}_0$. Then apply the filter operation m times (cf. (14)) and transform $\widehat{\mathbf{P}}_m$ back into $\mathbf{P}_m$. For this last transformation, FFT can be exploited effectively since the dimension of the *vector* $\widehat{\mathbf{P}}_m$ is $n\,2^m$.

Applying the filter in the frequency domain is also very easy since we just have to scale the coefficients by a constant factor and insert zeros. Depending on whether the number of vertices in the original polygon $\mathbf{P}_0 = (\mathbf{p}_0, \ldots, \mathbf{p}_{n-1})$ is even or odd, $\widehat{\mathbf{P}}_m$ takes the form

$$\widehat{\mathbf{P}}_m = \Big(2^m\,\hat{\mathbf{p}}_0, \ldots, 2^m\,\hat{\mathbf{p}}_{\frac{n}{2}-1}, 2^{m-1}\,\hat{\mathbf{p}}_{\frac{n}{2}}, \underbrace{0, \ldots, 0}_{(2^m-1)\,n-1}, 2^{m-1}\,\hat{\mathbf{p}}_{\frac{n}{2}}, 2^m\,\hat{\mathbf{p}}_{\frac{n}{2}+1}, \ldots, 2^m\,\hat{\mathbf{p}}_{n-1}\Big).$$

or

$$\widehat{\mathbf{P}}_m = \Big(2^m\,\hat{\mathbf{p}}_0, \ldots, 2^m\,\hat{\mathbf{p}}_{\frac{n-1}{2}}, \underbrace{0, \ldots, 0}_{(2^m-1)\,n}, 2^m\,\hat{\mathbf{p}}_{\frac{n+1}{2}}, \ldots, 2^m\,\hat{\mathbf{p}}_{n-1}\Big)$$

respectively. Since the number of non-vanishing coefficients in the frequency spectrum $\widehat{\mathbf{P}}_m$ is constant, the limit curve will obviously be the band limited interpolant. Fig. 4 shows an example for the application of this scheme.

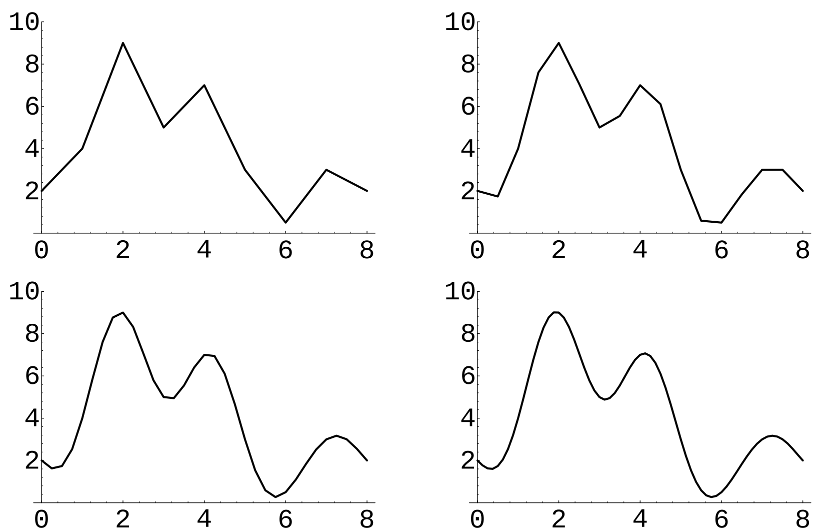

Figure 4: Iterative application of the C^∞-scheme.

References

[BBN87] H. Babovski / T. Beth / H. Neunzert / M. Schulz-Reese *Mathematische Methoden in der Systemtheorie: Fourieranalysis*, B. G. Teubner Stuttgart, 1987

[CDM91] A. Cavaretta / W. Dahmen / C. Micchelli, *Stationary Subdivision*, Memoirs of the AMS 93 (1991), pp. 1-186

[DD89] G. Deslauriers / S. Dubuc, *Symmetric Iterative Interpolation Processes*, Constr. Approx. 5 (1989), pp. 49–68

[Dau92] I. Daubechies, *Ten Lectures on Wavelets*, SIAM 1992

[DGL87] N. Dyn / J. Gregory / D. Levin, *A 4-Point Interpolatory Subdivision Scheme for Curve Design*, CAGD 4(1987), pp. 257–268

[DL94] N. Dyn / D. Levin, *Analysis of Asymptotically Equivalent Binary Subdivision Schemes*, J. Math. Anal. Appl. 193 (1994), pp.594–621

[Dyn91] N. Dyn, *Subdivision Schemes in Computer Aided Geometric Design*, Advances in Numerical Analysis II, Wavelets, Subdivisions and Radial Functions, W.A. Light ed., Oxford University Press, 1991, pp: 36-104.

[GY83] N. Geckinli / D. Yavuz, *Discrete Fourier Transformation and Its Applications to Power Spectra Estimation*, Elsevier Scientific Publishing Company 1983

[HV93] C. Herley / M. Vetterli, *Wavelets and recursive filter banks*, IEEE Trans. Sig. Proc. 41 (1993), pp. 2536–2556

[Kob95] L. Kobbelt, *Iterative Erzeugung glatter Interpolanten*, Thesis, Universität Karlsruhe, 1995

[Kob96] L. Kobbelt, *A Variational Approach to Subdivision*, CAGD 13(1996), pp. 743–761, Elsevier

[LR80] J. Lane / R. Riesenfeld, *A Theoretical Development for the Computer Generation and Display of Piecewise Polynomial Surfaces*, IEEE Trans. on Pattern Anal. and Mach. Int., 2 (1980), pp. 35–46

[OS89] A. Oppenheim / R. Schafer, *Discrete-Time Signal Processing*, Prentice Hall, 1989

[Qu92] R. Qu / J. Gregory, *A Subdivision Algorithm for Non-uniform B-splines*, Approximation Theory, Spline Functions and Applications, NATO ASI Series C: Mathematical and Physical Sciences, 356 (1992), pp. 423-436

[Rio92] O. Rioul, *Simple regularity criteria for subdivision schemes*, SIAM J. Math. Anal. 23 (1992), pp. 1544–1576

[War95] J. Warren, *Binary Subdivision Schemes for Functions over Irregular Knot Sequences*, to appear in Math. Meth in CAGD III, M. Daehlen, T. Lyche, L. Schumaker (eds.), 1995